Olympiad Number Theory Through Challenging Problems

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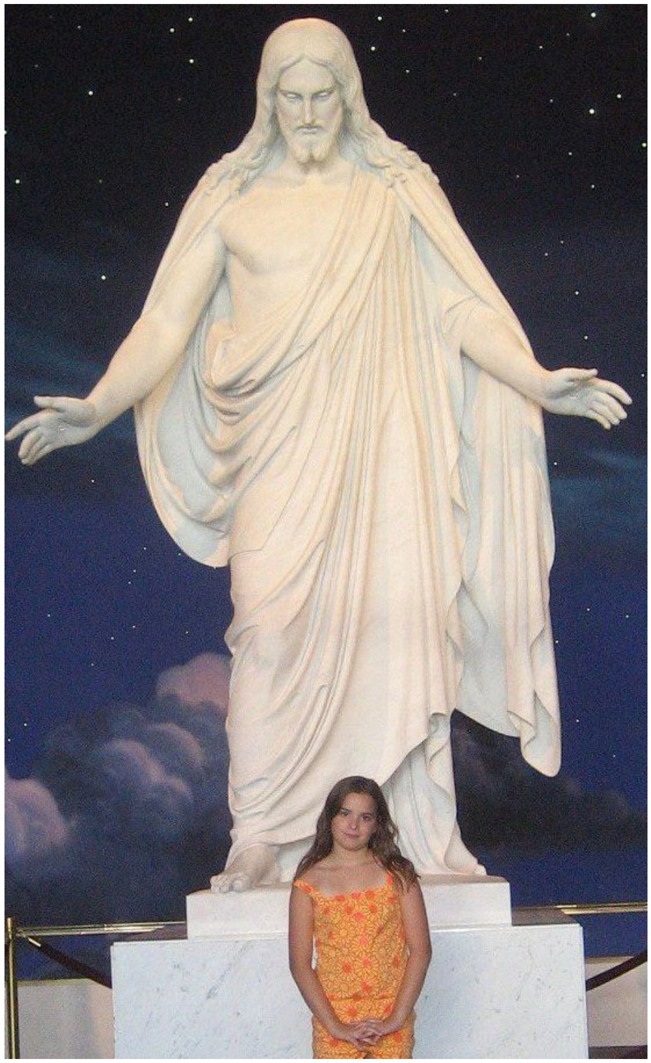
THIRD EDITION

Contents



For Cassie Stevens.
January 30th, 2000 to July 17th, 2013
"You never know how strong you are until being strong is the only choice you have."

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Divisibility

In this chapter, we will explore divisibility, the building block of number theory. This chapter will introduce many important concepts that will be used throughout the rest of the book. Divisibility is an extremely fundamental concept in number theory, and has applications including puzzles, encrypting messages, computer security, and many algorithms. An example is checking whether Universal Product Codes (UPC) or International Standard Book Number (ISBN) codes are legitimate.

Figure 1.1: An example of a UPC code.

In order for the 12 digit UPC code above to be legitimate, we order the digits $x_1, x_2, x_3, \cdots, x_{12}$. The expression

$$3x_1 + x_2 + 3x_3 + x_4 + 3x_5 + x_6 + 3x_7 + x_8 + 3x_9 + x_{10} + 3x_{11} + x_{12}$$

then must be divisible by 10. We indeed verify that the above code gives

$$0\times3+3\times1+6\times3+0\times1+0\times3+0\times1+2\times3+9\times1+1\times3+4\times1+5\times3+2\times1 = 60,$$

which is divisible by 10. Therefore the above UPC code is valid.

1.1 Euclidean and Division Algorithm

When the concept of division is first introduced in primary school, quotients and remainders are used. We begin with a simple picture that should be familiar to the reader, and we explore its relevance.

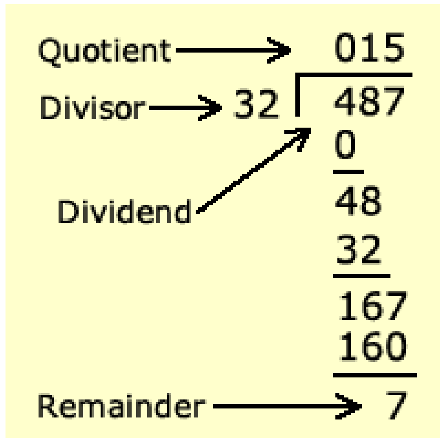

Figure 1.2: Division in primary school.
Source: CalculatorSoup

The process above used to divide 487 by 32 can be formalized through the division algorithm.

Theorem 1.1.1 (Division Algorithm). *For every integer pair a, b, there exists distinct integer quotient and remainders, q and r, that satisfy*

$$a = bq + r, \quad 0 \leq r < b$$

Proof. We will prove that this is true for when a and b are positive. The other cases when one or both of a and b are negative follow very similarly. There are two parts in this proof:

- Proving that for every pair (a, b) we can find a corresponding quotient and remainder.
- Proving that this quotient and remainder pair are unique.

For proving the existance of the quotient and remainder, given two integers a and b with varying q, consider the set

$$\{a - bq \text{ with } q \text{ an integer and } a - bq \geq 0\}.$$

By the well-ordering principle we know that this set must have a *minimum*, say when $q = q_1$. Clearly from the condition on the set, we must have $a - bq_1 = r \geq 0$. It now serves to prove that $a - bq_1 = r < b$. For the sake of contradiction, assume that $a - bq_1 \geq b$. However, then

$$a - b(q_1 + 1) \geq 0,$$

therefore it also should be a member of the above set. Furthermore,

$$a - b(q_1 + 1) < a - bq_1,$$

contradicting the minimality of q_1. Therefore, it is impossible for $a - bq_1 \geq b$, and we have $0 \leq a - bq_1 < b$.

The second part of this proof is to show that the quotient and remainder are unique. Assume for the sake of contradiction that a can be represented in two ways:

$$\begin{aligned} a = bq_1 + r_1 &= bq_2 + r_2 \\ b(q_1 - q_2) &= r_2 - r_1. \end{aligned}$$

This implies that $b \mid r_2 - r_1$. However,

$$b > r_2 - r_1 > -b$$

since $0 \leq r_1, r_2 < b$. Since $r_2 - r_1$ is a multiple of b, we must have $r_2 - r_1 = 0 \implies r_2 = r_1$ and $q_2 = q_1$. $\square$

For instance, when $a = 102$ and $b = 18$, applying the division algorithm gives $102 = 18 \times 5 + 12$, therefore $q = 5$ and $r = 12$. Furthermore, note that $\gcd(a, b) = \gcd(102, 18) = 6$ and $\gcd(b, r) = \gcd(18, 12) = 6$. This leads us to our next interesting result.

Theorem 1.1.2 (Euclid)**.** *For natural numbers a, b, we use the division algorithm to determine a quotient and remainder, q, r, such that $a = bq + r$. Then $\gcd(a, b) = \gcd(b, r)$.*

Proof. I claim that the set of common divisors between a and b is the same as the set of common divisors between b and r. If d is a common divisor of a and b, then since d divides both a and b, d divides all linear combinatinations of a and b. Therefore, $d \mid a - bq = r$, meaning that d is also a common divisor of b and r.

Conversely, if d is a common divisor of b and r, then d is a common divisor of all linear combinations of b and r, therefore, $d \mid bq + r = a$. Hence, d is also a common divisor of a and b.

We have established that the two sets of common divisors are equivalent, therefore, the greatest common divisor must be equivalent. $\square$

Corollary 1.1.1 (Euclidean Algorithm). *For two natural a, b, $a > b$, to find* $\gcd(a, b)$ *we use the division algorithm repeatedly*

$$\begin{aligned} a &= bq_1 + r_1 \\ b &= r_1 q_2 + r_2 \\ r_1 &= r_2 q_3 + r_3 \\ \cdots & \\ r_{n-2} &= r_{n-1} q_n + r_n \\ r_{n-1} &= r_n q_{n+1}. \end{aligned}$$

Then we have $\gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2) = \cdots = \gcd(r_{n-1}, r_n) = r_n$.

Proof. For each of the equations above, simply use Euclid's Theorem to arrive at the equality chain. □

Example 1.1.1. *Find* $\gcd(110, 490)$.

Solution.

$$\begin{aligned} 490 &= 110 \times 4 + 50 \\ 110 &= 50 \times 2 + \boxed{10} \\ 50 &= 10 \times 5. \end{aligned}$$

□

The division algorithm also works in $\mathbb{Q}[x]$, the set of polynomials with rational coefficients, and $\mathbb{R}[x]$, the set of all polynomials with real coefficients. For the sake of our study, we will only focus on $\mathbb{Q}[x]$. If $a(x)$ and $b(x)$ are two polynomials, then we can find a unique quotient and remainder polynomial, $q(x), r(x) \in \mathbb{Q}[x]$, such that

$$a(x) = b(x)q(x) + r(x), \quad \deg(r) < \deg(b) \text{ or } r(x) = 0.$$

We present a proof after an example. We can find $q(x)$ and $r(x)$ using long division for polynomials.

Example 1.1.2. *Calculate $q(x)$ and $r(x)$ such that $a(x) = b(x)q(x) + r(x)$ for $a(x) = x^4 + 3x^3 + 10$ and $b(x) = x^2 - x$.*

Solution. We begin by dividing the leading term of $a(x)$ by the leading term of $b(x)$: $\frac{x^4}{x^2} = x^2$. Therefore, we multiply $b(x)$ by x^2 and subtract the result from $a(x)$:

$$x^4 + 3x^3 + 10 = (x^2 - x)(x^2) + (4x^3 + 10).$$

Now, in order to get rid of the $4x^3$ term in the remainder, we have to divide this by the leading term of $b(x)$, x^2: $\frac{4x^3}{x^2} = 4x$. We add this to the quotient and subtract this multiplication from the remainder in order to get rid of the cubic term:

$$x^4 + 3x^3 + 10 = (x^2 - x)(x^2 + 4x) + (4x^2 + 10.)$$

One may be tempted to stop here, however, the remainder and $b(x)$ are both quadratic and we need $\deg(r(x)) < \deg(b(x))$. Therefore, in order to remove the quadratic term from the remainder, we divide this term, $4x^2$, by the leading term of $b(x)$, x^2: $\frac{4x^2}{x^2} = 4$. We then add this to the quotient, and subtract, in order to get

$$x^4 + 3x^3 + 10 = (x^2 - x)(x^2 + 4x + 4) + (4x + 10).$$

Therefore, $q(x) = x^2+4x+4$ and $r(x) = 4x+10$. We verify that indeed $\deg(r(x)) = 1 < \deg(b(x)) = 2$, therefore, we are finished.

Note: The numbers will not always come out as nicely as they did in the above expression, and we will occasionally have fractions. □

Theorem 1.1.3. *For two polynomials, $a(x), b(x) \in \mathbb{Q}[x]$, prove that there exists a unique quotient and remainder polynomial, $q(x)$ and $r(x)$, such that*

$$a(x) = b(x)q(x) + r(x), \quad deg(r) < deg(b) \text{ or } r(x) = 0.$$

Proof. For any two polynomials $a(x)$ and $b(x)$, we can find $q(x)$ and $r(x)$ such that $a(x) = b(x)q(x) + r(x)$ by repeating the procedure above. The main idea is to eliminate the leading term of $r(x)$ repeatedly, until $\deg(r(x)) < \deg(b(x))$.

- Divide the leading term of $a(x)$ by the leading term of $b(x)$ in order to obtain the polynomial $q_1(x)$. In the example above, we found $q_1(x) = \frac{x^4}{x^2} = x^2$ and $r_1(x) = 4x^3 + 10$. Then,

  $$a(x) = b(x)q_1(x) + r_1(x).$$

- Divide the leading term of $r_1(x)$ by the leading term of $b(x)$ in order to obtain the polynomial $q_2(x)$. In the example above, we found $q_2(x) = \frac{4x^3}{x^2} = 4x$. Then, add this quotient to $q_1(x)$ and subtract in order to find $r_2(x)$:

  $$a(x) = b(x)\left(q_1(x) + q_2(x)\right) + r_2(x).$$

  In the example above, $r_2(x) = 4x^2 + 10$.

- Repeat the above step of dividing the leading term of $r_j(x)$ by the leading term of $b(x)$ and adding this quotient to the previous quotients. So long as $\deg(r_j(x)) \geq \deg(b(x))$, this will decrease the degree of the remainder polynomial by eliminating its leading term. Stop once $\deg(r_j(x)) < \deg(b(x))$, at which point $q(x) = \sum_{i=1}^{j}(q_i(x))$ and $r(x) = r_j(x)$.

For the uniqueness part, note that if there exists distinct quotients $q_1(x)$, $q_2(x)$ and remainders $r_1(x), r_2(x)$ with $\deg(r_1(x)) < \deg(b(x))$ and $\deg(r_2(x)) < \deg(b(x))$ found through the division algorithm, we will arrive at a contradiction:

$$\begin{aligned} a(x) &= b(x)q_1(x) + r_1(x) \\ a(x) &= b(x)q_2(x) + r_2(x) \\ b(x)(q_1(x) - q_2(x)) &= r_2(x) - r_1(x). \end{aligned}$$

However, assuming that $q_1(x)$ and $q_2(x)$ are distinct, we have

$$\deg\left[b(x)(q_1(x) - q_2(x)\right] \geq \deg(b(x)).$$

On the other hand, since $\deg(r_1(x)) < \deg(b(x))$ and $\deg(r_2(x)) < \deg(b(x))$, we know that

$$\deg\left(r_2(x) - r_1(x)\right) < \deg(b(x)).$$

Therefore, it is impossible for the left hand side of the equation above to equal the right hand side since the degrees of the polynomials are different. □

Using the division algorithm for polynomials, we can extend Euclid's Algorithm for polynomials. Note that by convention, the greatest common divisor of two polynomials is chosen to be the **monic** polynomial of highest degree that divides both polynomials. The word monic means that the leading coefficient is 1. For instance, $\gcd(x^2 - 4, x - 2) = x - 2$.

Using the same reasoning we used for Euclid's theorem above, we can arrive at a similar theorem for polynomials.

Theorem 1.1.4. *If $a(x) = b(x)q(x) + r(x)$ with deg$(r(x)) <$ deg$(b(x))$, then*

$$\gcd(a(x), b(x)) = \gcd(b(x), r(x)).$$

Proof. We invoke the same method we used above by showing that the set of common divisors between $a(x)$ and $b(x)$ is the same as the set of common divisors between $b(x)$ and $r(x)$. □

Extending this method, we can calculate $\gcd(a(x), b(x))$:

$$\begin{aligned} a(x) &= b(x)q_1(x) + r_1(x) \\ b(x) &= r_1(x)q_2(x) + r_2(x) \\ r_1(x) &= r_2(x)q_3(x) + r_3(x) \\ &\cdots \\ r_{n-1}(x) &= r_n(x)q_{n+1}(x) + r_{n+1}(x) \\ r_n(x) &= r_{n+1}(x)q_{n+2}(x). \end{aligned}$$

Then we have

$$\gcd(a(x), b(x)) = \gcd(b(x), r_1(x)) = \gcd(r_1(x), r_2(x)) = \cdots = r_{n+1}(x),$$

the final non-zero remainder.

Example 1.1.3. *Find the greatest common divisor of $x^4 + x^3 - 4x^2 + x + 5$ and $x^3 + x^2 - 9x - 9$.*

Solution. Using polynomial division, we find that

$$x^4 + x^3 - 4x^2 + x + 5 = (x^3 + x^2 - 9x - 9)x + \left(5x^2 + 10x + 5\right).$$

Next, we have to divide $x^3 + x^2 - 9x - 9$ by $5x^2 + 10x + 5$. We find that

$$x^3 + x^2 - 9x - 9 \quad = \quad (5x^2 + 10x + 5)\left(\frac{x}{5} - \frac{1}{5}\right) + (-8x - 8).$$

Finally, we divide $5x^2 + 10x + 5$ by $-8x - 8$ and find that

$$5x^2 + 10x + 5 = (-8x - 8)\left[-\frac{5}{8}(x + 1)\right].$$

This is the final non-zero remainder. However, remembering that the greatest common divisor of two polynomials must be monic, we get rid of the $\frac{-5}{8}$ term and determine that $\gcd(x^4 + x^3 - 4x^2 + x + 5, x^3 + x^2 - 9x - 9) = \boxed{x + 1}$.

As a quick verification, we note that $x = -1$ is a root of both of the polynomials above. $\square$

We now move onto some contest style questions that involve the Euclidan Algorithm or the Division Algorithm.

Example 1.1.4 (Duke)**.** *What is the sum of all integers n such that n^2+2n+2 divides $n^3+4n^2+4n-14$?*

Solution. Using long division for polynomials, we find that

$$n^3 + 4n^2 + 4n - 14 = (n^2 + 2n + 2)(n + 2) + (-2n - 18).$$

In order for $n^2 + 2n + 2$ to divide $n^3 + 4n^2 + 4n - 14$, it must also divide the remainder:

$$(n^2 + 2n + 2) \mid (-2n - 18).$$

The only way that this is possible is either when $|-2n - 18| \geq |n^2 + 2n + 2|$ or when $-2n-18 = 0$. In the first case, this inequality only holds when $-4 \leq n \leq 4$. We test all n within this range, and determine that the values of n which work are $n = -4, -2, -1, 0, 1, 4$. In the second case, we additionally find that $n = -9$ works. Therefore, the sum is $\boxed{-11}$. □

Example 1.1.5 (AIME 1986)**.** *What is the largest positive integer n such that n^3+100 is divisible by $n+10$?*

Solution. Let

$$\begin{aligned} n^3 + 100 &= (n + 10)\left(n^2 + an + b\right) + c \\ &= n^3 + n^2(10 + a) + n(b + 10a) + 10b + c. \end{aligned}$$

Equating coefficients yields

$$\begin{cases} 10 + a = 0 \\ b + 10a = 0 \\ 10b + c = 100. \end{cases}$$

Solving this system yields $a = -10, b = 100$, and $c = -900$. Therefore, by the Euclidean Algorithm, we get

$$n + 10 = \gcd(n^3 + 100, n + 10) = \gcd(-900, n + 10) = \gcd(900, n + 10)$$

The maximum value for n is hence $n = \boxed{890}$. □

Example 1.1.6 (AIME 1985). *The numbers in the sequence 101, 104, 109, 116, ... are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.*

Solution. Since $d_n = \gcd\left(100 + n^2, 100 + (n+1)^2\right)$, d_n must divide the difference between these two, or $d_n \mid (100 + (n+1)^2) - (100 + n^2) = 2n + 1$. Therefore

$$d_n = \gcd(100 + n^2, 100 + (n+1)^2) = \gcd(n^2 + 100, 2n + 1).$$

Since $2n + 1$ will always be odd, 2 will never be a common factor, hence we can multiply $n^2 + 100$ by 4 without affecting the greatest common divisor:

$$\begin{aligned} d_n = \gcd(4n^2 + 400, 2n + 1) &= \gcd\left(4n^2 + 400 - (2n+1)(2n-1), 2n + 1\right) \\ &= \gcd\left(401, 2n + 1\right). \end{aligned}$$

Therefore, in order to maximize the value of d_n, we set $n = 200$ to give a greatest common divisor of $\boxed{401}$. □

The following theorem is very useful for problems involving exponents.

Theorem 1.1.5. *For natural numbers a, m, n, $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$.*

Outline. Note that by the Euclidean Algorithm, we have

$$\begin{aligned} \gcd(a^m - 1, a^n - 1) &= \gcd(a^m - 1 - a^{m-n}(a^n - 1), a^n - 1) \\ &= \gcd(a^{m-n} - 1, a^n - 1). \end{aligned}$$

We can continue to reduce the exponents using the Euclidean Algorithm, until we ultimately have $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$. □

Example 1.1.7. *Let the integers a_n and b_n be defined by the relationship*

$$a_n + b_n\sqrt{2} = \left(1 + \sqrt{2}\right)^n.$$

for all integers $n \geq 1$. Prove that $\gcd(a_n, b_n) = 1$ for all integers $n \geq 1$.

Solution. We use induction. For the base case, note that when $n = 1$, we have $a_1 = 1, b_1 = 1$, therefore, $\gcd(a_1, b_1) = 1$.

For the inductive hypothesis, we assume that it holds for $n = k$, therefore, when $a_k + b_k\sqrt{2} = \left(1 + \sqrt{2}\right)^k$, we have $\gcd(a_k, b_k) = 1$. We now show that it holds for $n = k + 1$. Note that

$$\begin{aligned} a_{k+1} + b_{k+1} &= \left(1+\sqrt{2}\right)^{k+1} \\ &= \left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)^{k} \\ &= \left(1+\sqrt{2}\right)\left(a_k + b_k\sqrt{2}\right) \\ &= (a_k + 2b_k) + \sqrt{2}\,(a_k + b_k)\,. \end{aligned}$$

Therefore, $a_{k+1} = a_k + 2b_k$ and $b_{k+1} = a_k + b_k$. It is now left to show that $\gcd(a_{k+1}, b_{k+1}) = 1$. Note that by the Euclidean Algorithm,

$$\gcd(a_k + 2b_k, a_k + b_k) = \gcd(b_k, a_k + b_k) = \gcd(b_k, a_k) = 1.$$

Therefore, by induction, we have shown that $n = k \implies n = k + 1$, and we are done. □

Example 1.1.8. *If p is an odd prime, and a, b are relatively prime positive integers, prove that*

$$\gcd\left(a + b, \frac{a^p + b^p}{a + b}\right) = 1 \text{ or } p.$$

Solution. We attempt to simplify the problem to the case when $b = 1$. Our goal is to now show that

$$\gcd(a + 1, \frac{a^p + 1}{a + 1}) = 1 \text{or } p.$$

Factoring gives

$$\frac{a^p + 1}{a + 1} = a^{p-1} - a^{p-2} + a^{p-3} - a^{p-4} + \cdots - a + 1.$$

In order to calculate $\gcd(a + 1, \frac{a^p+1}{a+1})$, we attempt to reduce the above expression mod $a + 1$. Using the fact that p is an odd prime, we know that $p - 1$ is even, therefore:

$$\begin{aligned} \frac{a^p + 1}{a + 1} &= a^{p-1} - a^{p-2} + \cdots + a^{2x} - a^{2x-1} + \cdots - a + 1 \\ &\equiv (-1)^{p-1} - (-1)^{p-2} + \cdots + (-1)^{2x} - (-1)^{2x-1} - a + 1 \pmod{a + 1} \\ &\equiv \underbrace{1 + 1 + \cdots + 1}_{\text{p terms}} \equiv p \pmod{a + 1}. \end{aligned}$$

Now, by the Euclidan Algorithm, we have

$$\gcd(a+1, \frac{a^p+1}{a+1}) = \gcd(a+1, p).$$

Since p is a prime, the above expression can only be equal to 1 or p, depending on a. We have now solved the problem for $b = 1$. We wish to generalize the method to any b.

Using a similar factorization as above, we have

$$\frac{a^p+b^p}{a+b} = a^{p-1} - a^{p-2}b + a^{p-3}b^2 - a^{p-4}b^3 + \cdots - ab^{p-2} + b^{p-1}.$$

In order to invoke the Euclidean Algorithm, we wish to evaluate this expression mod $a+b$. Using the fact that $a \equiv -b \pmod{a+b}$ and that $p-1$ is even, we can simplify as follows:

$$\begin{aligned} a^{p-1} - a^{p-2}b + a^{p-3}b^2 - \cdots + b^{p-1} &\equiv (-b)^{p-1} - (-b)^{p-2}\,b + (-b)^{p-3}\,b^2 + \cdots \\ &\equiv (-1)^{p-1}\left(\underbrace{b^{p-1} + b^{p-1} + \cdots + b^{p-1}}_{\text{p terms}}\right) \\ &\equiv pb^{p-1} \pmod{a+b}. \end{aligned}$$

Therefore, by the Euclidean Algorithm, we arrive at

$$\gcd\left(\frac{a^p+b^p}{a+b}, a+b\right) = \gcd(pb^{p-1}, a+b).$$

Now, in the problem statement, it was given that a and b are relatively prime. Hence, similarly, $\gcd(b, a+b) = 1$, and we can simplify the above expression further:

$$\gcd(pb^{p-1}, a+b) = \gcd(p, a+b) = 1 \text{ or } p.$$

$\square$

Example 1.1.9 (Japan 1996)**.** *Let m, n be relatively prime positive integers. Calculate* $\gcd(5^m + 7^m, 5^n + 7^n)$.

Solution. Without Loss Or Generality (WLOG), let $m > n$. Note that

$$5^m + 7^m = (5^n + 7^n)\left(5^{m-n} + 7^{m-n}\right) - 5^n 7^{m-n} - 5^{m-n} 7^n.$$

We now have two cases.

- If $m < 2n$, then factor out $5^{m-n}7^{m-n}$ from the right hand side of the above equation in order to get
$$5^m + 7^m = (5^n + 7^n)\left(5^{m-n} + 7^{m-n}\right) - 5^{m-n}7^{m-n}\left(5^{2n-m} + 7^{2n-m}\right).$$
Therefore, by the Euclidean Algorithm,
$$\begin{aligned}\gcd(5^m + 7^m, 5^n + 7^n) &= \gcd\left(5^{m-n}7^{m-n}(5^{2n-m} + 7^{2n-m}), 5^n + 7^n\right) \\ &= \gcd(5^{2n-m} + 7^{2n-m}, 5^n + 7^n).\end{aligned}$$
Since 5 and 7 both do not divide $5^n + 7^n$.

- If $m > 2n$, then factor out 5^n7^n from the right hand side of the first equation in order to get
$$5^m + 7^m = (5^n + 7^n)\left(5^{m-n} + 7^{m-n}\right) - 5^n7^n\left(5^{m-2n} + 7^{m-2n}\right).$$
Therefore, by the Euclidean Algorithm, and using the same logic as above,
$$\gcd(5^m + 7^m, 5^n + 7^n) = \gcd(5^n + 7^n, 5^{m-2n} + 7^{m-2n}).$$

Let $a_{m,n} = \gcd(5^m + 7^m, 5^n + 7^n)$ for simplicity. In summary from the two cases above, if $m < 2n$, then $a_{m,n} = a_{n,2n-m}$. On the other hand, if $m > 2n$, then $a_{m,n} = a_{n,m-2n}$.

If, for instance, we begin with $m = 12$ and $n = 5$, then the chain will go as follows:
$$a_{12,5} \to a_{2,5} \to a_{2,1} \to a_{0,1}.$$
Note that each step in the process decreases the sum of the two values, and furthermore, the parity of the sum remains the same at each step. Since m and n are relatively prime and the process is invariant mod 2, if $m + n$ is odd, trying out a few other cases will reveal that following this chain always give
$$a_{m,n} = a_{0,1} = \gcd(5^0 + 7^0, 5^1 + 7^1) = 2.$$
On the other hand, if for instance $m = 13$ and $n = 5$, then the chain will go as follows:
$$a_{13,5} \to a_{3,5} \to a_{3,1} \to a_{1,1}.$$
If $m + n$ is even, then we will always have
$$a_{m,n} = a_{1,1} = \gcd(5^1 + 7^1, 5^1 + 7^1) = 12.$$
In conclusion,
$$\gcd(5^m + 7^m, 5^n + 7^n) = \begin{cases} 12 & \text{if } 2 \mid m + n \\ 2 & \text{if } 2 \nmid m + n \end{cases}.$$
$\square$

1.2 Bezout's Identity

One of the immediate applications of the Euclidean Algorithm is Bezout's Identity (sometimes also called Bezout's Lemma).

Theorem 1.2.1 (Bezout's Identity)**.** *For a, b natural, there exist $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$.*

Proof. We present two proofs.

Euclidean Algorithm: Run the Euclidean Algorithm backwards.

$$\begin{aligned} \gcd(a,b) &= r_{n-2} - r_{n-1}q_n \\ &= r_{n-2} - (r_{n-3} - r_{n-2}q_{n-1})\, q_n \\ &= r_{n-2}(1 + q_n q_{n-1}) - r_{n-3}\,(q_n) \\ &= \cdots \\ &= ax + by. \end{aligned}$$

Where x and y are some combination of the quotients. The two variables run through at every step in the equation are:

$$(r_{n-2}, r_{n-1}) \to (r_{n-2}, r_{n-3}) \to (r_{n-4}, r_{n-3}) \cdots \to (b, r_1) \to (a, b).$$

Extremal Method: Consider the set

$$S = \{ax + by > 0 \text{ with } x, y \text{ integers}\}.$$

By the well-ordering principle, this set must have a minimum, say $d = \min(S)$. Since d is a member of the set, there exists integers x_1 and y_1 such that $d = ax_1 + by_1$. Now, we go about proving that $d = \gcd(a, b)$. To begin, we must show that d is a divisor of both a. By the division algorithm, say

$$a = dq + r, \quad 0 \le r < d.$$

Substituting $d = ax_1 + by_1$ into the above equation gives

$$a = d(ax_1 + by_1) + r \implies r = a(1 - dx_1) + b(-dy_1).$$

Therefore, if r is positive, then r is a member of the set S above. However, we know that $0 \le r < d$, contradicting the minimality of d. Hence, we must have $r = 0 \implies d \mid a$. Similarly, we can show that $d \mid b$. We have now shown that d is a common divisor of a and b.

It is now left to show that d is the *greatest* common divisor of a and b. Indeed, let d_1 be another common divisor of a and b. Therefore, d_1 also divides any linear combination of a and b, specifically $d_1 \mid ax_1 + by_1 = d$. Therefore, every common divisor of a and b divides d, therefore, $d = \gcd(a, b)$ and we are finished. □

Example 1.2.1. *Express* 5 *as a linear combination of* 45 *and* 65.

Solution. We use the Euclidean Algorithm in reverse. Using the Euclidean Algorithm on 45 and 65, we arrive at

$$\begin{aligned} 65 &= 45 \times 1 + 20 \\ 45 &= 20 \times 2 + 5 \\ 20 &= 5 \times 4. \end{aligned}$$

Therefore, we run the process in reverse to arrive at

$$\begin{aligned} 5 &= 45 - 20 \times 2 \\ &= 45 - (65 - 45 \times 1)2 \\ &= 45 \times 3 - 65 \times 2. \end{aligned}$$

□

Example 1.2.2. *Express* 10 *as a linear combination of* 110 *and* 380.

Solution. We again, use the Euclidean Algorithm to arrive at

$$\begin{aligned} 380 &= 110 \times 3 + 50 \\ 110 &= 50 \times 2 + 10 \\ 50 &= 10 \times 5. \end{aligned}$$

Now, running the Euclidean Algorithm in reverse gives us

$$\begin{aligned} 10 &= 110 - 50 \times 2 \\ &= 110 - (380 - 110 \times 3) \times 2 \\ &= 7 \times 110 - 2 \times 380. \end{aligned}$$

□

Example 1.2.3. *Express* 3 *as a linear combination of* 1011 *and* 11, 202.

Solution. We use the Euclidean Algorithm to arrive at

$$\begin{aligned} 11202 &= 1011 \times 11 + 81 \\ 1011 &= 81 \times 12 + 39 \\ 81 &= 39 \times 2 + 3 \\ 39 &= 3 \times 13. \end{aligned}$$

Now, runing the Euclidean Algorithm in reverse, we arrive at:

$$\begin{aligned} 3 &= 81 - 39 \times 2 \\ &= 81 - (1011 - 81 \times 12) \times 2 = 81 \times 25 - 1011 \times 2 \\ &= (11202 - 1011 \times 11) \times 25 - 1011 \times 2 = 11202 \times 25 - 1011 \times 277. \end{aligned}$$

□

Furthermore, Bezout's identity holds for any number of variables.

Theorem 1.2.2 (General Bezout's Identity)**.** *For integers $a_1, a_2, \cdots, a_n$, there exists integers $x_1, x_2, \cdots, x_n$ such that*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{i=1}^{n} a_ix_i = \gcd(a_1, a_2, \cdots, a_n).$$

Proof. Again, two methods can be used (either the Extremal Method or using induction). We show the induction proof for moving from 2 to 3 variables and challenge the reader to attempt using both methods to prove the general case. For 3 variables, we use 2 variable Bezout's twice:

$$\begin{aligned} \gcd(a_1, a_2, a_3) &= \gcd\left(a_1, \gcd(a_2, a_3)\right) \\ &= a_1x_1 + c\gcd(a_2, a_3) \\ &= a_1x_1 + c\left(x_2a_2 + x_3a_3\right) = a_1x_1 + a_2\left(cx_2\right) + a_3\left(cx_3\right) \end{aligned}$$

. □

Theorem 1.2.3 (Euclid's Lemma)**.** *If $a \mid bc$ and $\gcd(a, b) = 1$, prove that $a \mid c$.*

Proof. By Bezout's identity, $\gcd(a, b) = 1$ implies that there exist x, y such that

$$ax + by = 1.$$

Next, multiply this equation by c to arrive at

$$c(ax) + c(by) = c.$$

Finally, since $a \mid ac$ and $a \mid bc$, we have $a \mid ac(x) + bc(y) = c$. □

Bezout's identity for polynomials works the same exact way as it does for integers. Assume $f(x), g(x) \in \mathbb{Z}[x]$, then using Euclid's Algorithm, we can find $u(x), v(x) \in \mathbb{Q}[x]$ such that

$$f(x)u(x) + g(x)v(x) = \gcd(f(x), g(x)).$$

Here is an example for clarity.

Example 1.2.4. *Find polynomials* $u, v \in \mathbb{Q}[x]$ *such that*

$$(x^4 - 1)u(x) + (x^7 - 1)v(x) = (x - 1).$$

Solution. First off, we use Euclid's Algorithm on $x^4 - 1, x^7 - 1$. Notice that

$$\begin{aligned} x^7 - 1 &= (x^4 - 1)x^3 + x^3 - 1 \\ x^4 - 1 &= x(x^3 - 1) + x - 1 \\ x^3 - 1 &= (x - 1)(x^2 + x + 1). \end{aligned}$$

Therefore,

$$\begin{aligned} x - 1 &= x^4 - 1 - x(x^3 - 1) \\ &= x^4 - 1 - x\left[x^7 - 1 - \left(x^4 - 1\right)x^3\right] \\ &= \left(x^4 - 1\right) + x^4\left(x^4 - 1\right) - x\left(x^7 - 1\right) \\ &= \left(x^4 - 1\right)\left(x^4 + 1\right) - x\left(x^7 - 1\right). \end{aligned}$$

Therefore $u(x) = x^4 + 1, v(x) = -x$. □

The following example illustrates how to approach problems when the numbers are not as nice as above.

Example 1.2.5. *Find* $u, v \in \mathbb{Q}[x]$ *such that* $(2x^2 - 1)u(x) + (3x^3 - 1)v(x) = 1$.

Solution. By the division algorithm, note that

$$3x^3 - 1 = \left(2x^2 - 1\right)\left(\frac{3}{2}x\right) + \left(\frac{3}{2}x - 1\right).$$

Now, for the second step of the Euclidean Algorithm, we have to divide $2x^2 - 1$ by $\frac{3}{2}x - 1$. We start by dividing the leading terms: $\dfrac{2x^2}{\frac{3}{2}x} = \dfrac{4}{3}x$. Therefore, we have

$$2x^2 - 1 = \left(\frac{3}{2}x - 1\right)\left(\frac{4}{3}x\right) + \left(\frac{4}{3}x - 1\right).$$

Next, we divide the leading term of $r_1(x)$, $\frac{4}{3}x$, by the leading term of the dividend, $\frac{3}{2}x$: $\dfrac{\frac{4}{3}x}{\frac{3}{2}x} = \dfrac{8}{9}$. We add this to the quotient and get

$$2x^2 - 1 = \left(\frac{3}{2}x - 1\right)\left(\frac{4}{3}x + \frac{8}{9}\right) - \frac{1}{9}.$$

Running these steps in reverse, we see that

$$\begin{aligned}
-\frac{1}{9} &= (2x^2-1) - \left(\frac{4}{3}x + \frac{8}{9}\right)\left(\frac{3}{2}x - 1\right) \\
&= (2x^2-1) - \left(\frac{4}{3}x + \frac{8}{9}\right)\left[(3x^3-1) - (2x^2-1)\left(\frac{3}{2}x\right)\right] \\
&= (2x^2-1)\left[1 - \left(\frac{4}{3}x + \frac{8}{9}\right)\left(-\frac{3}{2}x\right)\right] + (3x^3-1)\left[-\left(\frac{4}{3}x + \frac{8}{9}\right)\right] \\
&= (2x^2-1)\left(2x^2 + \frac{4}{3}x + 1\right) + (3x^3-1)\left[-\left(\frac{4}{3}x + \frac{8}{9}\right)\right].
\end{aligned}$$

Finally, we have to multiply both sides by -9 in order to give 1 on the left hand side:

$$1 = (2x^2-1)(-18x^2 - 12x - 9) + (3x^3-1)(12x+8).$$

Therefore $u(x) = \boxed{-18x^2 - 12x - 9}$ and $v(x) = \boxed{12x + 8}$. □

Example 1.2.6. *Suppose you have a 5 litre jug and a 7 litre jug. We can perform any of the following moves:*

- *Fill a jug completely with water.*
- *Transfer water from one jug to another, stopping if the other jug is filled.*
- *Empty a jug of water.*

The goal is to end up with one jug having exactly 1 litre of water. How do we do this?

Solution. Note that at every stage, the jugs will contain a linear combination of 5 and 7 litres of water. We find that $1 = 5 \times 3 + 7 \times (-2)$, therefore, we want to fill the jug with 5 litres 3 times, and empty the one with 7 litres twice. In order to keep track of how much water we have in each step, we use an ordered pair (a, b), where a is the amount in the 5 litre jug and b is the amount in the 7 litre jug:

$$(0,0) \to (5,0) \to (0,5) \to (5,5) \to (3,7) \to (3,0) \to (0,3) \to (5,3) \to (1,7).$$

□

Example 1.2.7. *Use Bezout's identity to prove the theorem in Section 1.1,*

$$\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1.$$

Proof. Let $d = \gcd(a^m - 1, a^n - 1)$. Therefore, $a^m \equiv 1 \pmod{d}$ and $a^n \equiv 1 \pmod{d}$. By Bezout's identity, let $\gcd(m, n) = mx + ny$. Using the above two relations, we also have

$$a^{\gcd(m,n)} \equiv a^{mx+ny} \equiv a^{mx}a^{ny} \equiv 1 \pmod{d}.$$

Therefore, $d \mid a^{\gcd(m,n)} - 1$. We now show that $a^{\gcd(m,n)} - 1 \mid d$.

Since $\gcd(m, n) \mid m$, we have

$$a^{\gcd(m,n)} - 1 \mid a^m - 1.$$

We can similarly show that $a^{\gcd(m,n)} - 1 \mid a^n - 1$. Since $a^{\gcd(m,n)} - 1$ divides both $a^m - 1$ and $a^n - 1$, it must also divide their greatest common divisor:

$$a^{\gcd(m,n)} - 1 \mid \gcd(a^m - 1, a^n - 1) = d.$$

Since $d \mid a^{\gcd(m,n)} - 1$ and $a^{\gcd(m,n)} - 1 \mid d$, we must have $d = \gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$. □

Example 1.2.8. *(Putnam 2000) Prove that the expression*

$$\frac{\gcd(m, n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Solution. By Bezout's identity, there exist integers a and b such $\gcd(m, n) = am + bn$. Next, notice that

$$\frac{\gcd(m, n)}{n}\binom{n}{m} = \frac{am + bn}{n}\binom{n}{m} = \frac{am}{n}\binom{n}{m} + b\binom{n}{m}.$$

We must now prove that $\frac{am}{n}\binom{n}{m}$ is an integer. Note that

$$\frac{m}{n}\binom{n}{m} = \frac{m}{n}\left(\frac{n!}{m!(n-m)!}\right) = \frac{(n-1)!}{(m-1)!(n-m)!} = \binom{n-1}{m-1}.$$

Therefore,

$$\frac{\gcd(m, n)}{n}\binom{m}{n} = a\binom{m-1}{n-1} + b\binom{m}{n},$$

which is clearly an integer. □

1.3 Fundamental Theorem of Arithmetic

Next, we use Bezout's Identity to prove the Fundamental Theorem of Arithmetic, which, as the name suggests, is incredibly fundamental to mathematics.

Theorem 1.3.1. *(Fundamental Theorem of Arithmetic) Every integer $n \geq 2$ has a unique prime factorization.*

Proof. We divide this problem into two parts. The first part is showing that every integer $n \geq 2$ has a prime factorization. To do this, we use strong induction on n. To establish the base case, note that $n = 2, 3, 4$ all have a unique prime factorization. For the inductive hypothesis, assume that every integer $n < k$ has a prime factorization, and we show that $n = k$ then has a prime factorization.

If k is prime, then it has a prime factorization (itself). On the other hand, if k is composite, then let p be a prime divisor of k. We can now write $k = p\left(\frac{k}{p}\right)$. We know that $\frac{k}{p}$ can be written as the product of primes by the inductive hypothesis ($\frac{k}{p} < k$), therefore $k = p\left(\frac{k}{p}\right)$ similarly can be.

The second part of the problem is to prove uniqueness, for which we again use induction. The base cases of $n = 2, 3, 4$ all have unique prime factorizations. Assume that every integer $n < k$ has a unique prime factorization, and we prove that $n = k$ then must have a unique prime factorization. For the sake of contradiction, let k have two distinct prime factorizations, where repeated primes are allowed in the products:

$$n = p_1 p_2 p_3 \cdots p_i = q_1 q_2 q_3 \cdots q_j.$$

Note that we must have $p_1 \mid q_1 q_2 q_3 \cdots q_j$. By Euclid's Lemma (from Section 1.1), we know that we must have $p_1 \mid q_m$ for some integer m with $1 \leq m \leq j$. Therefore $p_1 = q_m$ since they are primes. Now, we can cancel this from both sides of the expression in order to get

$$\frac{n}{p_1} = \frac{n}{q_m} = p_2 p_3 \cdots p_i = q_1 q_2 \cdots q_{m-1} q_{m+1} \cdots q_j.$$

By the inductive hypothesis, $\frac{n}{p_i} = \frac{n}{q_m}$ has a unique prime factorization, therefore the two products above contain the same exact primes with the same multiplicity (although they may be slightly rearranged). Similarly, since $p_1 = q_m$, the two initial products are exactly identical, and n has a unique prime factorization. □

Theorem 1.3.2. *Let the prime factorizations of two integers a, b be*

$$\begin{aligned} a &= p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \\ b &= p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}. \end{aligned}$$

The exponents above can be zero and the p_i's are distinct. Then,

$$\begin{aligned} \gcd(a, b) &= p_1^{min(e_1, f_1)} p_2^{min(e_2, f_2)} \cdots p_k^{min(e_k, f_k)} \\ \textit{and } \operatorname{lcm}[a, b] &= p_1^{max(e_1, f_1)} p_2^{max(e_2, f_2)} \cdots p_k^{max(e_k, f_k)}. \end{aligned}$$

Corollary 1.3.1. *For $a, b \in \mathbb{Z}^+$, $\gcd(a, b) \operatorname{lcm}[a, b] = ab$.*

We now present several problems involving prime factorization, beginning with some more computational problems, and ending with some challenging olympiad problems.

Example 1.3.1. *(Classic) The cells in a jail are numbered from 1 to 100 and their doors are activated from a central button. The activation opens a closed door and closes an open door. Starting with all the doors closed the button is pressed 100 times. When it is pressed the k-th time the doors that are multiples of k are activated. Which doors will be open at the end?*

Solution. In order for a door to be open at the end, it will have to have been activated an odd number of times (since it initially was closed). For a given door d, it will be activated only when the button is pressed the k-th time if and only if k is a divisor of d. Therefore, we desire to find numbers that have an odd number of divisors.

Using the prime factorization of a in the theorem above, we calculate the number of prime divisors a has. In order to do this, we construct an arbitrary divisor of a. For each prime p_i in the prime factorization of a, we have $e_i + 1$ choices for the exponent. Therefore, we can see that the number of divisors of a is simply

$$\tau(a) = \prod_{i=1}^{k} (e_i + 1).$$

For this number to be odd, each exponent e_i must be even, implying that a must be a perfect square. Therefore, the doors which are open at the end are simply the doors that are perfect squares, namely $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$. $\square$

Example 1.3.2 (AIME 1998). *For how many values of k is 12^{12} the least common multiple of 6^6, 8^8, and k?*

Solution. We find the prime factorizations of these numbers. We have $12^{12} = 2^{24} \cdot 3^{12}$, $6^6 = 2^6 \cdot 3^6$ and $8^8 = 2^{24}$. Let $k = 2^{k_1}3^{k_2}$. Since $\operatorname{lcm}[2^6 \cdot 3^6, 2^{24}, k] = 2^{24} \cdot 3^{12}$, we must have $k_2 = 12$. On the other hand, since the second term above has 24 2's, there are no limitations on k_1 other than $0 \leq k_1 \leq 24$, giving 25 possibilities. Therefore, there are $25 \cdot 1 = \boxed{25}$ possible values of k. □

Example 1.3.3 (AIME 1987). *Let $[r, s]$ denote the least common multiple of positive integers r and s. Find the number of ordered triples a, b, c such that $[a, b] = 1000, [b, c] = 2000, [c, a] = 2000$.*

Solution. Notice that $1000 = 2^3 \times 5^3, 2000 = 2^4 \times 5^3$. Since we are working with least common multiples, set

$$a = 2^{a_1}5^{a_2}, b = 2^{b_1}5^{b_2}, c = 2^{c_1}5^{c_2}.$$

If a_1 or b_1 were at least 4, then 2^4 would divide $[a, b]$, therefore, this is impossible. On the other hand, $[b, c]$ and $[c, a]$ both are multiples of 2^4, therefore, we have $c_1 = 4$. Amongst a_1 and b_1, at least one of them must be 3 in order to have $2^3 \mid [a, b]$. Therefore, we have the pairs

$$(a_1, b_1, c_1) = (0, 3, 4), (1, 3, 4), (2, 3, 4), (3, 3, 4), (3, 2, 4), (3, 1, 4), (3, 0, 4).$$

for 7 in total.

Now, for the power of 5, in order to have all three of the least common multiples above be divisible by 5^3, at least two of the set a_2, b_2, c_2 must be 3. This gives us a total of 4 cases, when all of the numbers are 3 or when two of them are, and the third is less than 3:

$$(a_2, b_2, c_2) = (3, 3, 3), (3, 3, x), (3, x, 3), (x, 3, 3).$$

We know that $0 \leq x < 3$, therefore, there are 3 possibilities for each of the x's above. Therefore, there are a total of $3 \times 3 + 1 = 10$ possibilities for the powers of 5.

In conclusion, there is a total of $7 \times 10 = \boxed{70}$ ordered triples a, b, c which work. □

Example 1.3.4 (Canada 1970). *Given the polynomial*

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

with integer coefficients $a_1, a_2, \ldots, a_n$, *and given also that there exist four distinct integers* a, b, c *and* d *such that*

$$f(a) = f(b) = f(c) = f(d) = 5,$$

show that there is no integer k *such that* $f(k) = 8$.

Solution. Set $g(x) = f(x) - 5$. Since a, b, c, d are all roots of $g(x)$, we must have

$$g(x) = (x - a)(x - b)(x - c)(x - d)\, h(x)$$

for some $h(x) \in \mathbb{Z}[x]$. Let k be an integer such that $f(k) = 8$, giving $g(k) = f(k) - 5 = 3$. Using the factorization above, we find that

$$3 = (k - a)(k - b)(k - c)(k - d)\, h(x).$$

By the Fundamental Theorem of Arithmetic, we can only express 3 as the product of at most three distinct integers $(-3, 1, -1)$. Since $k - a, k - b, k - c, k - d$ are all distinct integers, we have too many terms in the product, leading to a contradiction. $\square$

Example 1.3.5. *Let* a, b, c *be positive integers. If* $\gcd(a, b, c) lcm[a, b, c] = abc$, *prove that* $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1$.

Solution. Consider a prime p that divides into at least one of a, b, c. I will show that it can divide into only **one** of the set a, b, c, hence, proving that a, b, c are pairwise relatively prime. We use the notation $p^k \parallel a$ to denote p^k fully dividing a, meaning that $p^k \mid a$, however, $p^{k+1} \nmid a$. Another form of this notation that will be seen later is $v_p(a) = k$.

For integers a_1, b_1, c_1, let $p^{a_1} \parallel a, p^{b_1} \parallel b, p^{c_1} \parallel c$. By the definition of greatest common divisor, $p^{\min(a_1,b_1,c_1)} \parallel \gcd(a, b, c)$. Similarly, by the definition of least common multiple, $p^{\max(a_1,b_1,c_1)} \parallel \text{lcm}[a, b, c]$. We assume WLOG that $a_1 \geq b_1 \geq c_1$; hence $\min(a_1, b_1, c_1) = c_1$ and $\max(a_1, b_1, c_1) = a_1$.

Therefore, the power of p which divides into $\gcd(a, b, c)\text{lcm}[a, b, c]$ is $\max(a_1, b_1, c_1) + \min(a_1, b_1, c_1) = a_1 + c_1$. Therefore,

$$p^{a_1+c_1} \parallel \gcd(a, b, c)\text{lcm}[a, b, c].$$

Note that, on the other hand,

$$p^{a_1+b_1+c_1} \parallel abc.$$

In order for this to be true, we must then have

$$a_1 + c_1 = a_1 + b_1 + c_1 \implies b_1 = 0.$$

By the WLOG assumption above, we had $a_1 \geq b_1 \geq c_1$. Since $b_1 = 0$, we must then have $c_1 = 0$. Therefore, p can only divide one of the set a, b, c. The same argument holds for any prime p that divides into the set $\{a, b, c\}$, therefore, a, b, c are pairwise relatively prime.

□

Example 1.3.6 (USAMO 1973)**.** *Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression.*

Solution. Assume for the sake of contradiction that three such distinct primes exist, and let these primes be $\sqrt[3]{p_1}, \sqrt[3]{p_2}, \sqrt[3]{p_3}$. By definition of an arithmetic sequence, set

$$\sqrt[3]{p_1} = a, \sqrt[3]{p_2} = a + kd, \sqrt[3]{p_3} = a + md, m > k.$$

Subtracting gives:

$$\begin{aligned} \sqrt[3]{p_2} - \sqrt[3]{p_1} &= kd \\ \sqrt[3]{p_3} - \sqrt[3]{p_1} &= md. \end{aligned}$$

Multiply the first equation by m and the second by k in order to equate the two

$$m\sqrt[3]{p_2} - m\sqrt[3]{p_1} = k\sqrt[3]{p_3} - k\sqrt[3]{p_1} = mkd.$$

Rearraging this equation, we get

$$m\sqrt[3]{p_2} - k\sqrt[3]{p_3} = (m-k)\sqrt[3]{p_1}. \tag{1.1}$$

Now, cubing this gives Using this equation and some rearrangement, we get:

$$m^3p_2 - 3\left(m^2p_2^{\frac{2}{3}}\right)\left(kp_3^{\frac{1}{3}}\right) + 3\left(mp_2^{\frac{1}{3}}\right)\left(k^2p_3^{\frac{2}{3}}\right) - k^3p_3 = (m-k)^3\, p_1.$$

Moving the integer terms over to the RHS and factoring out $3\left(mp_2^{\frac{1}{3}}\right)\left(kp_3^{\frac{1}{3}}\right)$ from the LHS gives

$$\left[3\left(mp_2^{\frac{1}{3}}\right)\left(kp_3^{\frac{1}{3}}\right)\right]\left(kp_3^{\frac{1}{3}} - mp_2^{\frac{1}{3}}\right) = (m-k)^3\, p_1 - m^3p_2 + k^3p_3.$$

From Equation 1.1, we know that $kp_3^{\frac{1}{3}} - mp_2^{\frac{1}{3}} = (k-m)\sqrt[3]{p_1}$. Therefore, substituting this into the above equation gives

$$3\left(mp_2^{\frac{1}{3}}\right)\left(kp_3^{\frac{1}{3}}\right)\left((k-m)\sqrt[3]{p_1}\right) = (m-k)^3 p_1 - m^3p_2 + k^3p_3.$$

Leaving only the cube roots on the left hand side gives

$$\sqrt[3]{p_1p_2p_3} = \frac{(m-k)^3 p_1 - m^3p_2 + k^3p_3}{3mk(k-m)}. \tag{1.2}$$

Lemma. If $\sqrt[3]{a}$ is rational, then $\sqrt[3]{a}$ is an integer.

Proof. Set $\sqrt[3]{a} = \frac{x}{y}$ where the fraction is in lowest terms, therefore $\gcd(x, y) = 1$. We desire to show that we must have the denominator, y, equal to 1. Cubing this equation and rearranging gives $ay^3 = x^3$. Assume for the sake of contradiction that y has a prime divisor, say p. If $p \mid y$ then we must also have $p \mid x$ from $ay^3 = x^3$. However, this contradicts $\gcd(x, y) = 1$. Therefore, it is impossible for y to have any prime divisors, and we must have $y = 1$, implying that $\sqrt[3]{a}$ is an integer. □

Since the RHS of Equation 1.2 is a rational number, the Lemma above implies that $\sqrt[3]{p_1p_2p_3}$ must be an integer. From the Fundamental Theorem of Arithmetic, this means that we must have $p_1 = p_2 = p_3$, contradiction.

□

1.4 Challenging Division Problems

In this section, I will show some of my favorite problems involving divisibility and concepts from this chapter. I highly recommend attempting these problems before reading the provided solutions.

These first few examples illustrate how to use inequalities and fractions with divisibility.

Example 1.4.1 (St. Petersburg 1996)**.** *Find all positive integers n such that*

$$3^{n-1} + 5^{n-1} \mid 3^n + 5^n.$$

Solution. Notice that $3^{n-1} + 5^{n-1}$ also divides 5 times itself:

$$3^{n-1} + 5^{n-1} \mid 5\left(3^{n-1} + 5^{n-1}\right) = 3^n + 2 \cdot 3^{n-1} + 5^n.$$

Subtracting the given equation from this gives us

$$\begin{aligned} 3^{n-1}+5^{n-1} &\mid \left(3^n+2\cdot 3^{n-1}+5^n\right)-\left(3^n+5^n\right) \\ \implies 3^{n-1}+5^{n-1} &\mid 2\cdot 3^{n-1}. \end{aligned}$$

However, for $n>1$, we have $3^{n-1}+5^{n-1}>2\cdot 3^{n-1}$, leading to the above divisibility being impossible. We then check that $n=1$ is the only possible solution. □

> **Example 1.4.2** (APMO 2002). *Find all pairs of positive integers a, b such that*
> $$\frac{a^2+b}{b^2-a} \text{ and } \frac{b^2+a}{a^2-b}$$
> *are both integers.*

Solution. For these conditions to be met, we must have

$$\begin{aligned} a^2+b\geq b^2-a &\qquad b^2+a\geq a^2-b \\ (a+b)(a-b+1)\geq 0 &\qquad (a+b)(b-a+1)\geq 0 \\ a\geq b-1 &\qquad b\geq a-1. \end{aligned}$$

For these two inequalities to be satisfied, we must have $a=b, b-1, b+1$. Notice that the cases $a=b-1$ and $a=b+1$ are identical because we can simply flip the pair after we finish. Therefore, we consider the below two cases.

Case 1: $a=b$.

The expression $\frac{a^2+a}{a^2-a}$ must then be an integer. Simplifying this expression shows that

$$\frac{a^2+a}{a^2-a}=\frac{a+1}{a-1}=1+\frac{2}{a-1}.$$

Therefore, $a-1$ must be a divisor of 2, giving $a=2,3$. This results in the solution pairs $(a,b)=(2,2),(3,3)$.

Case 2: $a=b-1$.

Note that

$$a^2+b=(b-1)^2+b=b^2-b+1=b^2-a.$$

Therefore, $\frac{a^2+b}{b^2-a}=1$. The only expression which we then have to check whether it is an integer is $\frac{b^2+a}{a^2-b}$. Expanding and simplifying shows:

$$\begin{aligned} \frac{b^2+a}{a^2-b} &= \frac{b^2+b-1}{(b-1)^2-b}=\frac{b^2+b-1}{b^2-3b+1} \\ &= 1+\frac{4b-2}{b^2-3b+1}. \end{aligned}$$

For $b \geq 7$, however, we then have $b^2 - 3b + 1 > 4b - 2$, meaning that the above expression cannot be an integer. We therefore test $b \in \{1, 2, 3, 4, 5, 6\}$ to see when $\frac{4b-2}{b^2-3b+1}$ is an integer, and find that $b = 1, 2, 3$ all work. However, when $b = 1$, this gives a non-positive value for a, therefore, we discard this solution. Therefore, the two solution pairs we find are $(a, b) = (1, 2), (2, 3)$.

The $a = b + 1$ case gives the permutations of the two above solutions, for a complete solution set of $(a, b) = (2, 2), (3, 3), (1, 2), (2, 3), (2, 1), (3, 2)$. $\square$

Example 1.4.3. *(1998 IMO) Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.*

Solution. We desire to simplify the above expression by cancelling some terms. If we multiply the divisor $xy^2 + y + 7$ by x and subtract this from y times the dividend, then the x^2y^2 and xy terms will cancel in the subtraction:

$$xy^2 + y + 7 \mid y\left(x^2y + x + y\right) - x\left(xy^2 + y + 7\right) = y^2 - 7x.$$

If $y^2 - 7x > 0$, then in order to have the above divisibility, we must have $xy^2 + y + 7 \leq y^2 - 7x$. However, for positive integers x, y, the LHS of the inequality is a lot larger, leading to this being impossible.

If $y^2 - 7x = 0$, then we have the solution pair $(x, y) = (7m^2, 7m)$ for positive integer m.

Finally, if $y^2 - 7x < 0$, then $7x - y^2 > 0$. For a positive integer d, let

$$\frac{7x - y^2}{xy^2 + y + 7} = d \implies 7x - y^2 = d(xy^2 + y + 7).$$

Expanding and isolating all the terms involving x to the LHS of the equation, we find that

$$x(7 - dy^2) = y^2 + dy + 7d.$$

Now, notice that the RHS must be positive since y and d are both positive integers. Therefore, similarly, the LHS must also be positive. However, for $y \geq 3$, we have $7 - dy^2 < 0$, therefore, we test $y = 1, 2$.

If $y = 1$, then we have

$$\begin{aligned} x(7 - d) &= 8d + 1 \\ \implies x &= \frac{8d + 1}{7 - d} = -8 + \frac{57}{7 - d}. \end{aligned}$$

The divisors of 57 are $1, 3, 19, 57$. We want both x and d to be positive integers, therefore, we have $7 - d = 1, 3$. In the first case, we have $x = -8 + \frac{57}{1} = 49$. In the second case, we have $x = -8 + \frac{57}{3} = 11$. Therefore, we arrive at the solutions $(x, y) = (11, 1), (49, 1)$.

If $y = 2$, then we have

$$x(7 - 4d) = 9d + 4.$$

However, since the LHS must be positive, we are forced to have $d = 1$, giving $x = \dfrac{9 \cdot 1 + 4}{7 - 4 \cdot 1} = \dfrac{13}{3}$, which is not an integer.

The solutions are hence $(x, y) = \boxed{(11, 1), (49, 1), (7m^2, 7m)}$. □

Example 1.4.4. *(1992 IMO) Find all integers a, b, c with $1 < a < b < c$ such that*

$$(a - 1)(b - 1)(c - 1)$$

divides $abc - 1$.

Solution. In order to simplify the above expression, we set $a = x+1, b = y+1, c = y + 1$. We then arrive at

$$xyz \mid (x + 1)(y + 1)(z + 1) - 1.$$

Expanding and simplifying give

$$\begin{aligned} xyz &\mid (xyz + xy + xz + yz + x + y + z + 1) - 1 \\ xyz &\mid xy + xz + yz + x + y + z. \end{aligned}$$

Finally, dividing the two expressions, we see that we must have

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} \in \mathbb{Z}.$$

From $1 < a < b < c$, we get $1 \le x < y < z$. The maximum possible value of this sum is when $x = 1, y = 2, z = 3$:

$$S = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} \le \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \times 2} + \frac{1}{1 \times 3} + \frac{1}{2 \times 3} = 2\frac{5}{6}.$$

Therefore, $S \in \{1, 2\}$. Furthermore, if we have $x, y, z \ge 3$, then the maximum possible value of S would be when $(x, y, z) = (3, 4, 5)$, giving $S = \frac{59}{60} \le 1$. Therefore, it is impossible for S to be an integer since it is less than 1. Since x is the smallest, we must have $x \in \{1, 2\}$. We have now greatly limited the possibilities for S and x, and we can do casework in order to find the corresponding values of y and z.

Case 1: If $x = 1$, then we get

$$\frac{1}{1} + \frac{1}{y} + \frac{1}{z} + \frac{1}{y} + \frac{1}{z} + \frac{1}{yz} = 1 \text{ or } 2.$$

In the case that $S = 1$, note that the LHS is greater than 1, therefore this is impossible. When $S = 2$, multiplying by yz and simplifying gives

$$2y + 2z + 1 = yz \implies (y-2)(z-2) = 5 \implies (y, z) = (3, 7)$$

In conclusion, we have the solution $(x, y, z) = (1, 3, 7)$.
Case 2: If $x = 2$, then we get

$$\frac{1}{2} + \frac{1}{y} + \frac{1}{z} + \frac{1}{2y} + \frac{1}{2z} + \frac{1}{yz} = 1 \text{ or } 2.$$

When $S = 1$, multiplying by $2yz$ and simplifying gives

$$3y + 3z + 2 = yz \implies (y-3)(z-3) = 11 \implies (y, z) = (4, 14).$$

When $S = 2$, multiplying by $2yz$ and simplifying gives

$$3y + 3z + 2 = 3yz.$$

However, since 2 is not a multiple of 3, this is impossible. In conclusion, we have the solution $(x, y, z) = (2, 4, 14)$.

In summary of the cases above, we found two solutions:

$$(x, y, z) = (2, 4, 14), (1, 3, 7) \implies (a, b, c) = (3, 5, 15), (2, 4, 8).$$

□

We conclude this chapter with a few interesting problems involving prime numbers and divisibility.

Example 1.4.5 (Iran 2005). *Let $n, p > 1$ be positive integers and p be prime. Given that $n \mid p - 1$ and $p \mid n^3 - 1$, prove that $4p - 3$ is a perfect square.*

Solution. Since $n \mid p - 1$, let $p - 1 = kn$ for some positive integer k, therefore $p = kn + 1$. This satisfies the first condition of the requirement. We now look at the second condition, which is $p \mid n^3 - 1 = (n-1)(n^2 + n + 1)$. Note that since $p = kn + 1$, we have $p \geq n - 1$, and because p is a prime, $\gcd(p, n - 1) = 1$:

$$p \mid (n-1)(n^2 + n + 1) \implies p = kn + 1 \mid n^2 + n + 1.$$

In order for this to be true, $kn+1 \leq n^2+n+1 \implies k \leq n+1$. Since $n^2+n+1 \mid k(n^2+n+1)$, we also have

$$\begin{aligned} p = kn+1 &\mid kn^2+kn+k \\ \implies kn+1 &\mid kn^2+kn+k-n(kn+1) = kn+k-n. \end{aligned}$$

Similarly, to have this divisibility, $kn+k-n \geq kn+1 \implies k \geq n+1$. However, above we found that $k \leq n+1$, therefore, $k = n+1$. Substituting this in for p gives $p = (n+1)n+1 = n^2+n+1$, giving

$$4p-3 = 4n^2+4n+4-3 = 4n^2+4n+1 = (2n+1)^2.$$

$\square$

Example 1.4.6 (David M. Bloom)**.** *Let p be a prime with $p > 5$, and let $S = \{p-n^2 | n \in \mathbb{N}, n^2 < p\}$. Prove that S contains two elements a and b such that $a|b$ and $1 < a < b$.*

Solution. We try a few examples. For $p = 23$, $S = \{23-1^2, 23-2^2, 23-3^2, 23-4^2\} = \{22, 19, 14, 7\}$ and $7 \mid 14$. For $p = 19$, $S = \{18, 15, 10, 3\}$ and $3 \mid 18$. For $p = 29$, $S = \{28, 25, 20, 13, 4\}$ and $4 \mid 20$. It appears as if the smallest element of S always divides another element.

Let k be so that $k^2 < p < (k+1)^2$. Therefore, $p-k^2$ is the smallest element in S.

If $p-k^2 = 1$, note that since $p > 5$, it must be an odd prime, therefore, k is even. Then, set $a = p-(k-1)^2 = 2k$ and $b = p-1^2 = k^2$. Since k is even, we have $a = 2k \mid b = k^2$.

In the case that $p-k^2 \neq 1$, then let $a = p-k^2$. Note that since $k^2 \equiv p \pmod{a}$, we also have $(k-a)^2 \equiv (a-k)^2 \equiv p \pmod{a}$. Therefore,

$$a \mid p-(k-a)^2 \text{ and } a \mid p-(a-k)^2.$$

It is now left to show that $p-(k-a)^2$ or $p-(a-k)^2 \in S$, depending on the sign of $k-a$.

If $k > a$, then let $n = k-a \in \mathbb{N}$. Furthermore, since $k^2 < p$, we also have $n^2 = (k-a)^2 < k^2 < p$, therefore $b = p-(k-a)^2 \in S$ and we have $a \mid b$.

If $k = a$, then we would have $k = p-k^2 \implies p = k(k+1)$, which is impossible for prime $p > 5$.

Finally, if $k < a$, then let $n = a-k \in \mathbb{N}$. Furthermore, we have

$$a = p-k^2 < (k+1)^2-k^2 = 2k+1 \implies a \leq 2k.$$

However, if $a = 2k$, then we would have $a = p - k^2 \implies p = k^2 + 2k = k(k+2)$, which is impossible for prime $p > 5$. Therefore, $a < 2k$ and we have $a - k < k$. Since $k^2 < p$, we also have $n^2 = (a-k)^2 < k^2 < p$, therefore, $b = p - (a-k)^2 \in S$ and $a \mid b$.

$\square$

Example 1.4.7. *(Iran 1998) Suppose that a and b are natural numbers such that*

$$p = \frac{b}{4}\sqrt{\frac{2a-b}{2a+b}}$$

is a prime number. Find all possible values of a, b, p.

Solution. If b is odd then $2a - b$ is odd so henceforth $\frac{b}{4}\sqrt{\frac{2a-b}{2a+b}}$ cannot be an integer. We consider two cases: $b = 4k$, or $b = 2m$ where m is odd.

Case 1: $b = 4k$.

$$\begin{aligned} p = k\sqrt{\frac{2a-4k}{2a+4k}} &= k\sqrt{\frac{a-2k}{a+2k}} \\ \left(\frac{p}{k}\right)^2 &= \frac{a-2k}{a+2k} \\ p^2a + 2p^2k &= k^2a - 2k^3 \\ a(k^2 - p^2) &= 2k(k^2 + p^2) \\ a &= \frac{2k(k^2+p^2)}{k^2 - p^2}. \end{aligned}$$

We know that a must be an integer. Consider the cases $p \mid k$ and $p \nmid k$. The first case gives us $k = dp$ for some positive integer d. Substituting this into the above equation gives

$$a = \frac{2dp(d^2p^2 + p^2)}{d^2p^2 - p^2} = \frac{2dp(d^2+1)}{d^2-1}.$$

We have $\gcd(d, d^2-1) = 1$ therefore we must have

$$\begin{aligned} \frac{2p(d^2+1)}{d^2-1} = 2p\left(1 + \frac{2}{d^2-1}\right) &\in \mathbb{Z} \\ \implies \frac{4p}{d^2-1} &\in \mathbb{Z}. \end{aligned}$$

When d is even we must have $(d^2-1) \mid p \implies p = d^2 - 1 = (d-1)(d+1)$ since p is a prime. The only way for this to be possible is when $d - 1 = 1$, which gives the solution $(p, d) = (3, 2)$. Substituting this into the formulas above gives $(a, b, p) = (20, 24, 3)$.

When d is odd, we find that $(p, d) = (2, 3)$. Substituting this into the formulas above gives $(a, b, p) = (15, 24, 2)$.

Now when $p \nmid k$, $a = \dfrac{2k(k^2+p^2)}{k^2-p^2}$ must again be an integer. From the Euclidean Algorithm, $\gcd(k, k^2-p^2) = \gcd(k, p^2) = 1$, therefore

$$\begin{aligned} \gcd(2k(k^2+p^2), k^2-p^2) &= \gcd(2k^2+2p^2, k^2-p^2) \\ &= \gcd(2k^2+2p^2-2(k^2-p^2), k^2-p^2) \\ &= \gcd(4p^2, k^2-p^2). \end{aligned}$$

By similar logic to above, $\gcd(p^2, k^2-p^2) = \gcd(p^2, k^2) = 1$, therefore we can further simplify:

$$\gcd(4p^2, k^2-p^2) = \gcd(4, k^2-p^2).$$

Since k^2-p^2 is the denominator of the fraction, we desire for the above expression to be equal to k^2-p^2. However, this gives $\begin{cases} k^2-p^2 = 1 \\ k^2-p^2 = 2 \\ k^2-p^2 = 4 \end{cases}$ of which none gives any positive solution for p.

In conclusion, we have found that $(a, b, p) = (20, 24, 3), (15, 24, 2)$.

Case 2: $b = 2m$ where m is odd.

$$p = \frac{m}{2}\sqrt{\frac{2a-2m}{2a+2m}}.$$

Using similar algebraic manipulation as before, we find that this is equivalent to

$$a = \frac{m(m^2+4p^2)}{m^2-4p^2}.$$

We again have two cases to consider: $p \mid m$ and $p \nmid m$. The first case $p \mid m$. Let $m = dp$ for some integer d. We arrive at

$$a = \frac{pd(p^2d^2+4p^2)}{p^2(d^2-4)} = \frac{pd(d^2+4)}{d^2-4}.$$

Remembering that since m is odd, d must also be odd, we have $\gcd(d, d^2-4) = \gcd(d, 4) = 1$ and $\gcd(d^2+4, d^2-4) = \gcd(8, d^2-4) = 1$. Therefore, in order for a to be an integer, $\dfrac{p}{d^2-4}$ must be an integer. Since the only divisors of a prime are 1 and itself,

$$d^2-4 = (d-2)(d+2) = p.$$

This is only possible when $d = 3$ which gives $p = 5$. Substituting these values into the above expressions gives the solution triple $(a, b, p) = (39, 30, 5)$.

On the other hand, when $p \nmid m$, $\gcd(m, m^2 - 4p^2) = \gcd(m, 4p^2) = 1$ since m is odd. Therefore, since a must be an integer, we have

$$\frac{m^2 + 4p^2}{m^2 - 4p^2} = 1 + \frac{8p^2}{m^2 - 4p^2} \in \mathbb{Z}.$$

Furthermore, we also have $\gcd(m^2 - 4p^2, 8p^2) = 1$ since m is odd, therefore we must have $m^2 - 4p^2 \mid 8$. This gives the cases: $\begin{cases} m^2 - 4p^2 = 1 \\ m^2 - 4p^2 = 2 \\ m^2 - 4p^2 = 4 \\ m^2 - 4p^2 = 8. \end{cases}$ which yields no valid integer solutions.

In conclusion, the three solutions are $(a, b, p) = (39, 30, 5), (20, 24, 3), (15, 24, 2)$. $\square$

1.5 Problems

1.1. Calculate $\gcd(301, 603)$.

1.2. Calculate $\gcd(153, 289)$.

1.3. Calculate $\gcd(133, 189)$.

1.4. The product of the greatest common factor and least common multiple of two positive numbers is 384. If one number is 8 more than the other number, compute the sum of two numbers.

1.5. Use long division on $a(x) = x^5 + 4x^4 + 2x$ and $b(x) = x^3 + 2x^2$ in order to calculate $q(x)$ and $r(x)$.

1.6. [AHSME 1990] For how many integers N between 1 and 1990 is the improper fraction $\frac{N^2+7}{N+4}$ not in lowest terms?

1.7. [IMO 1959] Prove that for natural n the fraction $\frac{21n+4}{14n+3}$ is irreducible.

1.8. Use the Euclidean Algorithm for polynomials to calculate $\gcd(x^4 - x^3, x^3 - x)$.

1.9. [AHSME 1988] If a and b are integers such that $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$, then what are the values of a and b?

1.10. Prove that any two consecutive terms in the Fibonacci sequence are relatively prime.

1.11. Prove that the sum and product of two positive relatively prime integers are themselves relatively prime.

1.12. Find integers x, y such that $5x + 97y = 1$.

1.13. Find integers x, y such that $1110x + 1011y = 3$.

1.14. Prove that there are no integers x, y such that $1691x + 1349y = 1$.

1.15. Find **all** integers x, y such that $5x + 13y = 1$.

1.16. Let $n \geq 2$ and k be positive integers. Prove that $(n-1)^2 \mid (n^k - 1)$ if and only if $(n-1) \mid k$.

1.17. Prove that $\sqrt{2}$ is irrational.

1.18. Prove that $\log_{10}(2)$ is irrational.

1.19. [AIME 2008] How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

1.20. Find polynomials $u, v \in \mathbb{Q}[x]$ such that

$$(5x^2 - 1)u(x) + (x^3 - 1)v(x) = 1.$$

1.21. Find $u, v \in \mathbb{Z}[x]$ such that $(x^8 - 1)u(x) + (x^5 - 1)v(x) = (x - 1)$.

1.22. For relatively prime naturals m, n, do there exist polynomials $u, v \in \mathbb{Q}[x]$ such that $(x^m - 1)u(x) + (x^n - 1)v(x) = (x - 1)$?

1.23. For positive integers $a, b, n > 1$, prove that

$$a^n - b^n \nmid a^n + b^n$$

1.24. [HMMT] Compute $\gcd(2002 + 2, 2002^2 + 2, 2002^3 + 2, \cdots)$.

1.25. Let n be a positive integer. Calculuate

$$\gcd\left(n! + 1, (n+1)!\right).$$

Note: This problem requires Wilson's theorem.

1.26. [PUMAC 2013] The greatest common divisor of $2^{30^{10}} - 2$ and $2^{30^{45}} - 2$ can be expressed in the form $2^x - 2$. Calculuate x.

1.27. [Poland 2004] Find all natural $n > 1$ for which value of the sum $2^2 + 3^2 + \cdots + n^2$ equals p^k where p is prime and k is natural.

1.28. Prove that if $m \neq n$, then

$$\gcd(a^{2^m} + 1, a^{2^n} + 1) = \begin{cases} 1 \text{ if a is even} \\ 2 \text{ if a is odd} \end{cases}.$$

1.29. Prove that for positive integers $a, b > 2$ we cannot have $2^b - 1 \mid 2^a + 1$.

1.30. [USAMO 2007] Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

2

Modular Arithmetic

2.1 Inverses

Definition 2.1.1. We say that the **inverse** of a number a modulo m when a and m are relatively prime is the number b such that $ab \equiv 1 \pmod{m}$.

Example. The inverse of 3 mod 4 is 3 because $3 \cdot 3 = 9 \equiv 1 \pmod{4}$. The inverse of 3 mod 5 is 2 because $3 \cdot 2 = 6 \equiv 1 \pmod{5}$.

The following theorem is incredibly important and helps us to prove Euler's Totient Theorem and the existence of an inverse. Make sure that you understand the proof and theorem as we will be using it down the road.

Theorem 2.1.1. *Let a and m be relatively prime positive integers. Let the set of positive integers relatively prime to m and less than m be $R = \{a_1, a_2, \cdots, a_{\phi(m)}\}$. Prove that $S = \{aa_1, aa_2, aa_3, \cdots, aa_{\phi(m)}\}$ is the same as R when reduced mod m.*

Proof. Notice that every element of S is relatively prime to m. Also R and S have the same number of elements. Because of this, if we can prove that no two elements of S are congruent mod m we would be done. However

$$aa_x \equiv aa_y \pmod{m} \implies a(a_x - a_y) \equiv 0 \pmod{m} \implies a_x \equiv a_y \pmod{m}$$

which happens only when $x = y$ therefore the elements of S are distinct mod m and we are done. $\square$

Theorem 2.1.2. *When* $\gcd(a, m) = 1$, *a always has a distinct inverse mod*

m.

Proof. We notice that $1 \in R$ where we define $R = \{a_1, a_2, \cdots, a_{\phi(m)}\}$ to be the same as above. This must be the same mod m as an element in $\{aa_1, aa_2, \cdots, aa_{\phi(m)}\}$ by Theorem 1 henceforth there exists some a_x such that $aa_x \equiv 1 \pmod{m}$. □

Corollary 2.1.1. *The equation $ax \equiv b \pmod{m}$ always has a solution when $\gcd(a, m) = 1$.*

Proof. Set $x \equiv a^{-1}b \pmod{m}$. □

Example. Find the inverse of 9 mod 82.

Solution. Notice that $9 \cdot 9 \equiv -1 \pmod{82}$ therefore $9 \cdot (-9) \equiv 1 \pmod{82}$. The inverse of 9 mod 82 is hence $82 - 9 = 73$. □

Example 2.1.1. *Let m and n be positive integers posessing the following property: the equation*

$$\gcd(11k - 1, m) = \gcd(11k - 1, n)$$

holds for all positive integers k. Prove that $m = 11^r n$ for some integer r.

Solution. Define $v_p(a)$ to be the number of times that the prime p occurs in the prime factorization of a [1]. The given statement is equivalent to proving that $v_p(m) = v_p(n)$ when $p \neq 11$ is a prime. To prove this, assume on the contrary that WLOG we have

$$v_p(m) > v_p(n)$$

Write $m = p^a b, n = p^c d$ where b and d are relatively prime to p. We have $a > c$.

By Theorem 2 we know that there exists a solution for k such that $11k \equiv 1 \pmod{p^a}$. However, we now have

$$p^a \mid \gcd(11k - 1, m)$$

but $p^a \mid \gcd(11k - 1, n)$ implies that $p^a \mid n$ contradicting $a > c$. We are done. □

[1] We explore this function more in depth later

Example 2.1.2. *Let a and b be two relatively prime positive integers, and consider the arithmetic progression $a, a+b, a+2b, a+3b, \cdots$. Prove that there are infinitely many pairwise relatively prime terms in the arithmetic progression.*

Solution. We use induction. The base case is trivial. Assume that we have a set with m elements that are all relatively prime. Let this set be $S = \{a+k_1b, a+k_2b, \cdots, a+k_mb\}$. Let the set $\{p_1, p_2, \cdots, p_n\}$ be the set of all distinct prime divisors of elements of S. I claim that we can construct a new element. Let

$$a + xb \equiv 1 \pmod{p_1 \cdot p_2 \cdot \cdots \cdot p_n}$$

We know that there exists a solution in x to this equation which we let be $x = k_{m+1}$. Since $\gcd(a + k_{m+1}b, a + k_ib) = 1$, we have constructed a set with size $m+1$ and we are done. □

2.2 Chinese Remainder Theorem

Theorem 2.2.1 (Chinese Remainder Theorem)**.** *The system of linear congruences*

$$\begin{cases} x \equiv a_1 \pmod{b_1}, \\ x \equiv a_2 \pmod{b_2}, \\ \cdots \\ x \equiv a_n \pmod{b_n}, \end{cases}$$

where $b_1, b_2, \cdots, b_n$ are pairwise relatively prime (aka $\gcd(b_i, b_j) = 1$ iff $i \neq j$) has one distinct solution for x modulo $b_1b_2\cdots b_n$.

Proof. We use induction. I start with proving that for the case

$$\begin{cases} x \equiv a_1 \pmod{b_1}, \\ x \equiv a_2 \pmod{b_2}, \end{cases}$$

there exists a unique solution mod b_1b_2. To do so, consider the set of numbers

$$S = \{kb_1 + a_1, 0 \leq k \leq b_2 - 1\}.$$

By Corollary 1 it follows that the equation $kb_1 + a_1 \equiv a_2 \pmod{b_2}$ has a distinct solution in k. We have shown the unique existence of a solution to the above system of linear congruences.

Assume there is a solution for $n = k$ and I prove that there is a solution for $n = k + 1$. Let the following equation have solution $x \equiv z \pmod{b_1 b_2 \cdots b_k}$ by the inductive hypothesis:

$$\begin{cases} x \equiv a_1 \pmod{b_1} \\ x \equiv a_2 \pmod{b_2} \\ \cdots \\ x \equiv a_k \pmod{b_k}. \end{cases}$$

Therefore to find the solutions to the $k+1$ congruences it is the same as finding the solution to

$$\begin{cases} x \equiv z \pmod{b_1 b_2 \cdots b_k} \\ x \equiv a_{k+1} \pmod{b_{k+1}}. \end{cases}$$

For this we can use the exact same work we used to prove the base case along with noting that from $\gcd(b_{k+1}, b_i) = 1$ for $i \in \{1, 2, \cdots, k\}$, we have $\gcd(b_{k+1}, b_1 b_2 \cdots b_k) = 1$. $\square$

Example 2.2.1. *Find the solution to the linear congruence*

$$\begin{cases} x \equiv 3 \pmod{5}, \\ x \equiv 4 \pmod{11}. \end{cases}$$

Solution. Notice that we may write x in the form $5k + 3$ and $11m + 4$.

$$x = 5k + 3 = 11m + 4$$

Taking this equation mod 5 we arrive at $11m + 4 \equiv 3 \pmod{5} \implies m \equiv -1 \pmod{5}$. We substitute $m = 5m_1 - 1$ to give us $x = 11(5m_1 - 1) + 4 = 55m_1 - 7$.

Therefore $x \equiv 48 \pmod{55}$ which means $\boxed{x = 55k + 48}$ for some integer k. $\square$

Example 2.2.2 (AIME II 2012)**.** *For a positive integer p, define the positive integer n to be p-safe if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is $3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 23, \ldots$. Find the number of positive integers less than or equal to $10,000$ which are simultaneously 7-safe, 11-safe, and 13-safe.*

Solution. We notice that if x is 7-safe, 11-safe, and 13-safe then we must have

$$\begin{cases} x \equiv 3, 4 \pmod 7 \\ x \equiv 3, 4, 5, 6, 7, 8 \pmod{11} \\ x \equiv 3, 4, 5, 6, 7, 8, 9, 10 \pmod{13} \end{cases}$$

By Chinese Remainder Solution this renders solutions mod 1001. We have 2 choices for the value of x mod 7, 6 choices for the value of x mod 11 and 8 choices for the value of x mod 13. Therefore, we have $2 \cdot 6 \cdot 8 = 96$ total solutions mod 1001.

We consider the number of solutions in the set

$$\{1, 2, \cdots, 1001\}, \{1002, \cdots, 2002\}, \{2003, \cdots, 3003\}, \cdots, \{9009, \cdots, 10010\}.$$

From above there are $96 \cdot 10 = 960$ total solutions. However we must subtract the solutions in the set $\{10, 001; 10, 002; \cdots; 10, 010\}$.

We notice that only $x = 10, 006$ and $x = 10, 007$ satisfy $x \equiv 3, 4 \pmod 7$.

	$10,006$	$10,007$
$\pmod 7$	3	4
$\pmod{11}$	7	8
$\pmod{13}$	9	10

These values are arrived from noting that $10, 006 \equiv -4 \pmod{7 \cdot 11 \cdot 13}$ and $10, 007 \equiv -3 \pmod{7 \cdot 11 \cdot 13}$. Therefore $x = 10, 006$ and $x = 10, 007$ are the two values we must subtract off.

In conclusion we have $960 - 2 = \boxed{958}$ solutions. $\square$

Example 2.2.3. *Consider a number line consisting of all positive integers greater than 7. A hole punch traverses the number line, starting frrom 7 and working its way up. It checks each positive integer n and punches it if and only if $\binom{n}{7}$ is divisible by 12. (Here $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.) As the hole punch checks more and more numbers, the fraction of checked numbers that are punched approaches a limiting number ρ. If ρ can be written in the form $\frac{m}{n}$, where m and n are positive integers, find $m + n$.*

Solution. Note that

$$\binom{n}{7} = \frac{n!}{(n-7)!7!} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2^4 \cdot 3^2 \cdot 5 \cdot 7}.$$

In order for this to be divisible by $12 = 2^2 \cdot 3$, the numerator must be divisible by $2^6 \cdot 3^3$. (We don't care about the 5 or the 7; by the Pigeonhole Principle these will be canceled out by factors in the numerator anyway.) Therefore we wish to find all values of n such that

$$2^6 \cdot 3^3 \,|\, n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6).$$

We start by focusing on the factors of 3, as these are easiest to deal with. By the Pigeonhole Principle, the expression must be divisible by $3^2 = 9$. Now, if $n \equiv 0, 1, 2, 3, 4, 5, \text{ or } 6 \pmod 9$, one of these seven integers will be a multiple of 9 as well as a multiple of 3, and so in this case the expression is divisible by 27. (Another possibility is if the numbers n, $n-3$, and $n-6$ are all divisible by 3, but it is easy to see that this case has already been accounted for.)

Now, we have to determine when the product is divisible by 2^6. If n is even, then each of $n, n-2, n-4, n-6$ is divisible by 2, and in addition exactly two of those numbers must be divisible by 4. Therefore the divisibility is sure. Otherwise, n is odd, and $n-1, n-3, n-5$ are divisible by 2.

- If $n-3$ is the only number divisible by 4, then in order for the product to be divisible by 2^6 it must also be divisible by 16. Therefore $n \equiv 3 \pmod{16}$ in this case.

- If $n-1$ and $n-5$ are both divisible by 4, then in order for the product to be divisible by 2^6 one of these numbers must also be divisible by 8. Therefore $n \equiv 1, 5 \pmod 8 \implies n \equiv 1, 5, 9, 13 \pmod{16}$.

Pooling all our information together, we see that $\binom{n}{7}$ is divisible by 12 iff n is such that

$$\begin{cases} n \equiv 0, 1, 2, 3, 4, 5, 6 & \pmod 9, \\ n \equiv 0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14 & \pmod{16}. \end{cases}$$

There are 7 possibilities modulo 9 and 13 possibilities modulo 16, so by CRT there exist $7 \times 13 = 91$ solutions modulo $9 \times 16 = 144$. Therefore, as more and more numbers n are checked, the probability that $\binom{n}{7}$ is divisible by 12 approaches $\frac{91}{144}$. The requested answer is $91 + 144 = \boxed{235}$.

$\square$

Example 2.2.4 (Austin Shapiro)**.** *Call a lattice point "visible" if the greatest common divisor of its coordinates is* 1*. Prove that there exists a* 100×100 *square on the board none of whose points are visible.*

Solution. Let the points on the grid be of the form

$$(x, y) = (a + m, b + n), \qquad 99 \geq m, n \geq 0.$$

We are going to use the Chinese Remainder Theorem to have every single term have a common divisor among the two coordinates. For the remainder of the problem assume that the sequence $\{p_j\}$ is a sequence of distinct prime numbers. Let $a \equiv 0 \left(\bmod \prod_{i=1}^{100} p_i\right)$. Then let

$$\begin{cases} b & \equiv 0 \pmod{p_1} \\ b+1 & \equiv 0 \pmod{p_2} \\ \cdots \\ b+99 & \equiv 0 \pmod{p_{100}}. \end{cases}$$

We find that repeating this process with letting $a+1 \equiv 0 \left(\bmod \prod_{i=101}^{200} p_i\right)$ and defining similarly

$$\begin{cases} b \equiv 0 \pmod{p_{101}} \\ b+1 \equiv 0 \pmod{p_{102}} \\ \cdots \\ b+99 \equiv 0 \pmod{p_{200}} \end{cases}$$

gives us the following:

$$\begin{cases} a & \equiv 0 \left(\bmod \prod_{i=1}^{100} p_i\right) \\ a+1 & \equiv 0 \left(\bmod \prod_{i=101}^{200} p_i\right) \\ \cdots \\ a+99 & \equiv 0 \left(\bmod \prod_{i=9901}^{10000} p_i\right) \end{cases} \quad \text{and} \quad \begin{cases} b & \equiv 0 \left(\bmod \prod_{i=0}^{99} p_{100i+1}\right) \\ b+1 & \equiv 0 \left(\bmod \prod_{i=0}^{99} p_{100i+2}\right) \\ \cdots \\ b+99 & \equiv 0 \left(\bmod \prod_{i=1}^{100} p_{100i}\right). \end{cases}$$

This notation looks quite intimidating; take a moment to realize what it is saying. It is letting each $a+k$ be divisible by 100 distinct primes, then letting b be divisible by the first of these primes, $b+1$ be divisible by the second of these primes and so forth. This is precisely what we did in our first two examples above. By CRT we know that a solution exists, therefore we have proven the existence of a 100×100 grid.

□

Motivation. This problem requires a great deal of insight. When I solved this problem, my first step was to think about completing a 1×100 square as we did above. Then you have to think of how to extend this method to a 2×100 square and then generalizing the method all the way up to a 100×100 square. Notice that our construction is not special for 100, we can generalize this method to a $x \times x$ square! □

2.3 Euler's Totient Theorem and Fermat's Little Theorem

Theorem 2.3.1 (Euler's Totient Theorem)**.** *For a relatively prime to m, we have $a^{\phi(m)} \equiv 1 \pmod{m}$.*

Proof. Using Theorem 2.1.1 the sets $\{a_1, a_2, \cdots, a_{\phi(m)}\}$ and $\{aa_1, aa_2, \cdots, aa_{\phi(m)}\}$ are the same mod m. Therefore, the products of each set must be the same mod m

$$a^{\phi(m)} a_1 a_2 \cdots a_{\phi(m)} \equiv a_1 a_2 \cdots a_{\phi(m)} \pmod{m} \implies a^{\phi(m)} \equiv 1 \pmod{m}.$$

□

Corollary 2.3.1 (Fermat's Little Theorem)**.** *For a relatively prime to a prime p, we have $a^{p-1} \equiv 1 \pmod{p}$.*

Proof. Trivial. □

Example. Find $2^{98} \pmod{33}$.

We do this problem in two different ways.

Solution. Notice that we may not directly use Fermats Little Theorem because 33 is not prime. However, we may use Fermats little theorem in a neat way: Notice that $2^2 \equiv 1 \pmod 3$ and $2^{10} \equiv 1 \pmod{11}$ from Fermats Little Theorem. We will find $2^{98} \pmod 3$ and $2^{98} \pmod{11}$ then combine the results to find $2^{98} \pmod{33}$.

$$2^{98} \equiv (2^2)^{49} \equiv 1^{49} \equiv 1 \pmod 3,$$
$$2^{98} \equiv \left[(2^{10})^9\right]\left(2^8\right) \equiv 2^8 \equiv 256 \equiv 3 \pmod{11}.$$

Let $x = 2^{98}$. Therefore, $x \equiv 1 \pmod 3$ and $x \equiv 3 \pmod{11}$. Inspection gives us $x \equiv \boxed{25} \pmod{33}$. □

OR

Solution. We will use Euler's Totient Theorem. Notice that $\phi(33) = 33(1-\frac{1}{3})(1-\frac{1}{11}) = 20$, therefore $2^{20} \equiv 1 \pmod{33}$. Now, notice that

$$x = 2^{98} = (2^{20})^5 2^{-2} \equiv 4^{-1} \pmod{33}.$$

Therefore, since $x \equiv 4^{-1} \pmod{33} \implies 4x \equiv 1 \pmod{33}$. Using quick analysis, $x \equiv 25 \pmod{33}$ solves this hence $2^{98} \equiv \boxed{25} \pmod{33}$. Our answer matches our answer above so we feel fairly confident in it. □

Example 2.3.1 (Brilliant.org)**.** *For how many integer values of i, $1 \le i \le 1000$, does there exist an integer j, $1 \le j \le 1000$, such that i is a divisor of $2^j - 1$?*

Solution. For i even it is clear that we can't have $i|(2^j-1)$. For i odd let $j = \phi(i)$ to give us $2^{\phi(i)} - 1 \equiv 0 \pmod i$. Since $\phi(i) < 1000$ it follows that for i odd there is always a value of j and hence the answer is $\frac{1000}{2} = \boxed{500}$. □

Example 2.3.2 (Brilliant.org)**.** *How many prime numbers p are there such that $29^p + 1$ is a multiple of p?*

Solution. When $p = 29$ we have $29 \nmid 29^p + 1$. Therefore we have $\gcd(p, 29) = 1$. By Fermat's Little Theorem

$$29^p + 1 \equiv 29 + 1 \equiv 0 \pmod p.$$

Therefore it follows that $p \mid 30$ and hence $p = 2, 3, 5$ for a total of $\boxed{3}$ numbers. □

Example 2.3.3 (AIME 1983)**.** *Let* $a_n = 6^n + 8^n$. *Determine the remainder on dividing* a_{83} *by* 49*.*

Solution. Notice that $\phi(49) = 42$. Therefore, $6^{84} \equiv 1 \pmod{49}$ and $8^{84} \equiv 1 \pmod{49}$.

$$\begin{aligned} 6^{83} + 8^{83} &\equiv (6^{84})(6^{-1}) + (8^{84})(8^{-1}) \pmod{49} \\ &\equiv 6^{-1} + 8^{-1} \pmod{49} \end{aligned}$$

Via expanding both sides out we find:

$$\begin{aligned} 6^{-1} + 8^{-1} &\equiv (6+8)6^{-1}8^{-1} \pmod{49} \\ &\equiv (14)\,48^{-1} \pmod{49} \\ &\equiv (14)\,(-1) \pmod{49} \\ &\equiv 35 \pmod{49} \end{aligned}$$

□

Tip 2.3.1. When dealing with $a^{\phi n-1} \pmod{n}$ it is often easier to compute $a^{-1} \pmod{n}$ then to compute $a^{\phi n-1} \pmod{n}$ directly.

Example 2.3.4 (All Russian MO 2000)**.** *Evaluate the sum*

$$\left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \cdots + \left\lfloor \frac{2^{1000}}{3} \right\rfloor.$$

Solution. Note that we have

$$2^x \equiv \begin{cases} 1 \pmod{3} & \text{when } x \text{ is even,} \\ 2 \pmod{3} & \text{when } x \text{ is odd.} \end{cases}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{1000} \left\lfloor \frac{2^n}{3} \right\rfloor = 0 + \sum_{n=1}^{500} \left(\left\lfloor \frac{2^{2n-1}}{3} \right\rfloor + \left\lfloor \frac{2^{2n}}{3} \right\rfloor \right) = \sum_{n=1}^{500} \left(\frac{2^{2n-1}-2}{3} + \frac{2^{2n}-1}{3} \right) \\ = \frac{1}{3} \sum_{n=1}^{500} (2^{2n-1} + 2^{2n} - 1) = \frac{1}{3} \sum_{n=1}^{1000} 2^n - 500 = \boxed{\frac{1}{3}(2^{1001}-2) - 500}. \end{aligned}$$

□

Tip 2.3.2. When working with floor functions, try to find a way to make the fractions turn into integers.

Example 2.3.5 (HMMT 2009)**.** *Find the last two digits of* 1032^{1032}. *Express your answer as a two-digit number.*

Solution. $\begin{cases} 1032^{1032} \equiv 0 \pmod 4 \\ 1032^{1032} \equiv 7^{1032} \equiv (-1)^{516} \equiv 1 \pmod{25} \end{cases} \implies 1032^{1032} \equiv \boxed{76}$ $\pmod{100}$ □

Tip 2.3.3. When we have to calculuate $a \pmod{100}$ it is often more helpful to find $\begin{cases} a \pmod 4 \\ a \pmod{25} \end{cases}$ and then using the Chinese Remainder Theorem to find $a \pmod{100}$. We can do similar methods when dealing with $a \pmod{1000}$.

Example 2.3.6 (Senior Hanoi Open MO 2006)**.** *Calculate the last three digits of* $2005^{11} + 2005^{12} + \cdots + 2005^{2006}$.

Solution. By reducing the expression modulo 1000, it remains to find the last three digits of the somewhat-less-daunting expression

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006} \equiv 5^{11} + 5^{12} + \cdots + 5^{2006} \pmod{1000}.$$

Notice that $5^{11} + 5^{12} + \cdots + 5^{2006} \equiv 0 \pmod{125}$.

Next, we want $5^{11} + 5^{12} + \cdots + 5^{2006} \pmod 8$. Notice that $5^2 \equiv 1 \pmod 8$ and therefore $5^{2k} \equiv 1 \pmod 8$ and $5^{2k+1} \equiv 5 \pmod 8$. Henceforth

$$5^{11} + 5^{12} + \cdots + 5^{2006} \equiv \frac{1996}{2}(1+5) \equiv 4 \pmod 8$$

.

Therefore $5^{11} + 5^{12} + \cdots + 5^{2006} \equiv \boxed{500} \pmod{1000}$.

□

Example 2.3.7 (PuMAC[a]). *Calculate the last* 3 *digits of* $2008^{2007^{2006^{\cdots^{2^1}}}}$.

[a]Princeton University Mathematics Competition

Solution. To begin we notice $2008^{2007^{2006^{\cdots^{2^1}}}} \equiv 0 \pmod 8$.

Next, we notice via Euler's Totient:

$$2008^{2007^{2006^{\cdots^{2^1}}}} \equiv 2008^{2007^{2006^{\cdots^{2^1}}} \pmod{\phi(125)}} \pmod{125}$$

Now notice

$$2007^{2006^{\cdots^{2^1}}} \equiv 7^{2006^{\cdots^{2^1}}} \pmod{100} \equiv 1 \pmod{100}$$

since $7^4 \equiv 1 \pmod{100}$. Therefore $2008^{2007^{2006^{\cdots^{2^1}}}} \equiv 2008^1 \equiv 8 \pmod{125}$. Henceforth

$$2008^{2007^{2006^{\cdots^{2^1}}}} \equiv \boxed{8} \pmod{1000}.$$

□

Tip 2.3.4. When a and n are relatively prime we have

$$a^b \equiv a^{b \pmod{\phi(n)}} \pmod n$$

It then suffices to calculate $b \pmod{\phi(n)}$.

Example 2.3.8 (PuMAC 2008). *Define* $f(x) = x^{x^{x^x}}$. *Find the last two digits of* $f(17) + f(18) + f(19) + f(20)$.

Solution. We compute each individual term seperately.

- $f(20) \equiv 0 \pmod{100}$
- $\begin{cases} f(19) \equiv (-1)^{19^{19^{19}}} \equiv -1[4] \\ f(19) \equiv 19^{19^{19^{19}} \pmod{\phi 25}} \equiv 19^{-1} \equiv 4[25] \end{cases} \implies f(19) \equiv 79[100]$
- $\begin{cases} 18^{18^{18^{18}}} \equiv 0[4] \\ 18^4 \equiv 1[25] \implies 18^{18^{18^{18}}} \equiv 1 \pmod{25} \end{cases} \implies f(18) \equiv 76[100]$

- $$\begin{cases} f(17) \equiv 1 \pmod{4} \end{cases}$$
$$\begin{cases} f(17) \equiv 17^{17^{17^{17}} \pmod{20}} \pmod{25} \\ \phi(20) = 8 \implies 17^{17^{17}} \equiv 17 \pmod{20} \\ f(17) \equiv 17^{17} \pmod{25} \\ 17^{16} \equiv (17^2)^8 \equiv ((14)^2)^4 \equiv \left((-4)^2\right)^2 \equiv 16^2 \equiv 6 \pmod{25} \\ \implies f(17) \equiv (17)(6) \equiv 2 \pmod{25} \end{cases}$$

$$\longrightarrow f(17) \equiv 77 \pmod{100}$$

Therefore

$$f(17) + f(18) + f(19) + f(20) \equiv 77 + 76 + 79 + 0 \equiv 232 \equiv \boxed{32} \pmod{100}$$

□

Example 2.3.9 (AoPS)**.** *Show that for $c \in \mathbb{Z}$ and a prime p, the congruence $x^x \equiv c \pmod{p}$ has a solution.*

Solution. Suppose that x satisfies the congruences

$$\begin{cases} x & \equiv c \pmod{p}, \\ x & \equiv 1 \pmod{p-1}. \end{cases}$$

Then by Fermat's Little Theorem we have

$$x^x \equiv x^{x \pmod{p-1}} \equiv x^1 \equiv c \pmod{p}.$$

All that remains is to show that there actually exists a number with these properties, but since $(p, p-1) = 1$, there must exist such a solution by Chinese Remainder Theorem. ■ □

Example 2.3.10 (Balkan)**.** *Let n be a positive integer with $n \geq 3$. Show that*

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

Solution. Note that $1989 = 3^2 \times 13 \times 17$. Therefore, we handle each case separately.

- First, we prove that
$$n^{n^{n^n}} \equiv n^{n^n} \pmod 9$$
for all n. If $3|n$ then we are done. Otherwise, since $\phi(9) = 6$, the problem reduces itself to proving that
$$n^{n^n} \equiv n^n \pmod 6$$
This is not hard to show true. It is obvious that $n^{n^n} \equiv n^n \pmod 2$, and (as long as n is not divisible by 3) showing the congruence modulo 3 is equivalent to showing that $n^n \equiv n \pmod{\phi(3)}$, which is trivial since n and n^n have the same parity.

- Next, we prove that
$$n^{n^{n^n}} \equiv n^{n^n} \pmod{13}$$
This is a very similar process. If $13|n$ we are done; otherwise, the problem is equivalent to showing that
$$n^{n^n} \equiv n^n \pmod{12}$$
We have already shown that these two expressions are congruent modulo 3, and showing that they are equivalent modulo 4 is equivalent to showing that $n^n \equiv n \pmod{\phi(4)}$, and since $\phi(4) = 2$ this is trivial. Therefore they are congruent modulo 12 and this second case is complete.

- Finally, we prove that
$$n^{n^{n^n}} \equiv n^{n^n} \pmod{17}$$
, which once again is quite similar. If $17|n$ we are done, otherwise the problem reduces itself to proving that
$$n^{n^n} \equiv n^n \pmod{16}$$
If $2|n$ in this case then the problem is solved; otherwise, it reduces itself again to proving that $n^n \equiv n \pmod 8$. This is the most difficult case of this

problem, but it is still not hard. Note that since n is odd, $n^2 \equiv 1 \pmod 8$. Therefore $n^{n-1} \equiv 1 \pmod 8$ and $n^n \equiv n \pmod 8$. Through this, we have covered all possibilities, and so we are done with the third case.

We have shown that the two quantities are equivalent modulo 9, 13, and 17, so by Chinese Remainder Theorem they must be equivalent modulo $9 \times 13 \times 17 = 1989$, as desired. $\square$

Example 2.3.11 (Canada 2003)**.** *Find the last* 3 *digits of* $2003^{2002^{2001}}$.

Solution. Taking the number directly mod 1000 doesn't give much (try it out and see!) Therefore we are going to do it in two different parts: Find the value mod 8 and find the value mod 125 then combine the two values. First off note that $\phi(8) = 4$ and $\phi(125) = 100$.

$$\begin{cases} 2003^{2002^{2001}} & \equiv 3^{2002^{2001}} \pmod 8 \\ 2002^{2001} & \equiv 0 \pmod{\phi(8)} \end{cases} \implies 3^{2002^{2001}} \equiv 3^0 \equiv 1 \pmod 8$$

Now notice that $2003^{2002^{2001}} \equiv 3^{2002^{2001}} \pmod{125}$. To find this we must find $2002^{2001} \equiv 2^{2001}$ (mod $\phi(125) = 100$). We split this up into finding $2^{2001} \pmod 4$ and $2^{2001} \pmod{25}$:

$$\begin{cases} 2^{2001} & \equiv 0 \pmod 4 \\ 2^{2001} = 2^1(2^{20})^{100} = 2^1(2^{\phi(25)})^{100} & \equiv 2 \pmod{25} \end{cases} \implies x \equiv 52 \pmod{100}$$

Therefore we have $3^{2002^{2001}} \equiv 3^{52} \pmod{125}$. It is a tedious work but as this problem appeared on a mathematical olympiad stage it should be done by hand.

$$\begin{aligned} 3^{52} &\equiv (3^5)^{10}(3^2) \pmod{125} \\ &\equiv (-7)^{10}(9) \pmod{125} \equiv (49)^5(9) \pmod{125} \\ &\equiv \left(49^2\right)^2(49)(9) \pmod{125} \equiv (26^2)(49)(9) \pmod{125} \\ &\equiv (51)(49)(9) \pmod{125} \equiv -9 \equiv 116 \pmod{125} \end{aligned}$$

Let $x = 2003^{2002^{2001}}$.

$$\begin{cases} x \equiv 1 \pmod 8 & \implies x = 8k+1 \\ x \equiv 116 \pmod{125} & \implies x = 125m + 116 \end{cases}.$$

Therefore, $x = 125m + 116 = 8k + 1$. Taking mod 8 of this equation we arrive at $5m + 4 \equiv 1 \pmod 8$ or $5m \equiv 5 \pmod 8$ hence $m \equiv 1 \pmod 8$. Henceforth we have $m = 8m_1 + 1$ or

$$x = 125(8m_1 + 1) + 116 = 1000m_1 + 241.$$

Therefore $x \equiv \boxed{241} \pmod{1000}$.

$\square$

Tips. *These are very helpful tips for dealing with problems involving exponentation.*

- *When dealing with* $a^{\phi n-1} \pmod n$ *it is often easier to compute* $a^{-1} \pmod n$ *then to compute* $a^{\phi n-1} \pmod n$ *directly.*
- *When we have to calculuate* $a \pmod{100}$ *it is often more helpful to find* $\begin{cases} a \pmod 4 \\ a \pmod{25} \end{cases}$ *and then using the Chinese Remainder Theorem to find* $a \pmod{100}$. *We can do similar methods when dealing with* $a \pmod{1000}$.
- *When* a *and* n *are relatively prime we have*

$$a^b \equiv a^{b \pmod{\phi(n)}} \pmod n$$

It then suffices to calculuate $b \pmod{\phi(n)}$.

Example 2.3.12 (Balkan MO 1999)**.** *Let* $p > 2$ *be a prime number such that* $3|(p-2)$. *Let*

$$S = \{y^2 - x^3 - 1 | 0 \le x, y \le p - 1 \cap x, y \in \mathbb{Z}\}$$

Prove that there are at most p *elements of* S *divisible by* p.

Solution. Despite the intimidiating notation, all that it is saying for S is that S is the set of $y^2 - x^3 - 1$ when x and y are positive integers and $0 \le x, y \le p - 1$ (essentially meaning x, y are reduced mod p).

Lemma. When $a \not\equiv b \pmod p$ we have $a^3 \not\equiv b^3 \pmod p$.

Proof. Assuming to the contrary that for some a, b with $a \not\equiv b \pmod{p}$ we have $a^3 \equiv b^3 \pmod{p}$. Since $\frac{p-2}{3}$ is an integer raising both sides to that power gives us

$$\begin{aligned}(a^3)^{\frac{p-2}{3}} &\equiv (b^3)^{\frac{p-2}{3}} \pmod{p}\\ a^{p-2} &\equiv b^{p-2} \pmod{p}\\ a^{-1} &\equiv b^{-1} \pmod{p}\\ a &\equiv b \pmod{p}\end{aligned}$$

With the last step following from the fact that every element has a unique inverse mod p.

□

We now count how many ways we have $y^2 - x^3 - 1 \equiv 0 \pmod{p}$. Notice that we must have $x^3 \equiv y^2 - 1 \pmod{p}$. For each value of y there is at most one value of x which corresponds to it. Since there are a total of p values of y there is at most p pairs (x, y) such that $y^2 - x^3 - 1 \equiv 0 \pmod{p}$. □

Motivation. The motivation behind this solution was trying a simple case. When $p = 5$ we notice that we have:

$x \pmod{5}$	$x^2 \pmod{5}$	$x^3 \pmod{5}$
0	0	0
1	1	1
2	4	3
3	4	2
4	1	4

We need for the difference of an x^2 and y^3 to be 1. We notice that for each value of x^2 there can only be one value of y^3 that goes along with it. Then we go on to prove the general case. □

Example 2.3.13 (USAMO 1991). *Show that, for any fixed integer $n \geq 1$, the sequence*

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \pmod{n}$$

is eventually constant.

[The tower of exponents is defined by $a_1 = 2$, $a_{i+1} = 2^{a_i}$. Also $a_i \pmod{n}$ means the remainder which results from dividing a_i by n.]

Solution. First, define the recursive sequence $\{a_k\}$ such that $a_1 = 2$ and $a_{i+1} = 2^{a_i}$ for each $i \geq 1$. It is clear that the sequence of a_i's maps to the sequence of integers described in the original problem.

In addition, for each positive integer n, consider a recursive sequence $\{b_k\}$ such that b_1 is the largest odd integer that evenly divides n and for each $i \geq 1$, b_{i+1} is the largest odd integer that evenly divides $\phi(b_i)$. For example, when $n = 62$, $b_1 = 31$, $b_2 = 15$, and $b_3 = 1$. Note that it is obvious that at some positive integer i we have $b_i = 1$, since the sequence of b_i's is monotonically decreasing.

Now consider a term of the sequence of towers of 2 with a sufficient number of 2s, say a_m. We wish to have for all sufficiently large m:

$$a_{m+1} \equiv a_m \pmod{n}$$

Note that we can write n in the form $2^{k_1}b_1$, where k_1 is a nonnegative integer. By Chinese Remainder Theorem, it suffices to check

$$\begin{cases} a_{m+1} \equiv a_m \pmod{2^{k_1}}, \\ a_{m+1} \equiv a_m \pmod{b_1}. \end{cases}$$

Since m is sufficiently large, $a_{m+1} \equiv a_m \equiv 0 \pmod{2^{k_1}}$. All that remains is to check if $a_{m+1} \equiv a_m \pmod{b_1}$. By Euler's Totient Theorem, since $\gcd(2, b_1) = 1$ we have $2^{\phi(b_1)} \equiv 1 \pmod{b_1}$, so

$$a_m = 2^{a_{m-1}} \equiv 2^{a_{m-1} \bmod \phi(b_1)} \pmod{b_1}.$$

Therefore, we now want $a_m \equiv a_{m-1} \pmod{\phi(b_1)}$. Since $\phi(b_1) = 2^{k_2}b_2$ we must check

$$\begin{cases} a_m \equiv a_{m-1} \pmod{2^{k_2}}, \\ a_m \equiv a_{m-1} \pmod{b_2}. \end{cases}$$

Because m is sufficiently large

$$a_m \equiv a_{m-1} \equiv 0 \pmod{2^{k_2}},$$

and so it suffices to check if $a_m \equiv a_{m-1} \pmod{b_2}$. We can continue this method all the way down to the integer i such that $b_i = 1$ at which point the equation $a_m \equiv a_{m-1} \equiv 0 \pmod{b_i}$ is clearly true so hence we have arrived at a true statement. Therefore $a_{m+1} \equiv a_m \pmod{n}$ for sufficiently large m and we are done. $\square$

Motivation. The choice for the b_i sequence may be a bit confusing therefore I try my best to explain the motivation here.

When wishing to prove $a_{m+1} \equiv a_m \pmod{n}$ by Euler's Totient we try to see if we can have $a_m \equiv a_{m-1} \pmod{\phi(n)}$. If $\phi(n)$ was odd then we could continue on this method until we have $\phi\left(\phi(\cdots\phi(n))\right) = 1$ at which point since we have chosen m to be sufficiently large we would get a true statement.

Thankfully we patch the arugment by considering the largest odd divisor of $\phi(n)$ and using Chinese Remainder Theorem. That is where the idea for the b_i sequence comes from. □

Example 2.3.14 (ISL 2005 N6)**.** *Let a, b be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers n. Prove that $a = b$.*

Solution. I desire to prove that for all primes $b \equiv a \pmod{p}$. Fix a to be a constant value and I construct a way to find the corresponding value of n which tells us that $b \equiv a \pmod{p}$. Notice that the following conditions are the same:

$$(a^n + n) \mid (b^n + n) \iff (a^n + n) \mid (b^n - a^n).$$

We set

$$\begin{cases} n \equiv -a \pmod{p}, \\ n \equiv 1 \pmod{p-1}. \end{cases}$$

Because $n \equiv 1 \pmod{p-1}$ we have $a^n \equiv a \pmod{p}$ via Fermat's Little Theorem. Also, because $n \equiv -a \pmod{p}$ we have $p|(a^n + n) \implies p|(b^n - a^n)$ However, $b^n \equiv b \pmod{p}$ and $a^n \equiv a \pmod{p}$ from $n \equiv 1 \pmod{p-1}$ therefore $b \equiv a \pmod{p}$ for all primes p hence $b = a$.

□

Motivation. The way I solved this problem was consider the case when $a = 1$ at first. In this case we must have $(n+1) \mid b^n + n$. We desire to prove $b \equiv 1 \pmod{p}$ for all primes p. We notice that $b^n + n \equiv b^n - 1 \pmod{n+1}$. If we have $p|(n+1)$ and $b^n \equiv b \pmod{p}$ then we would be done. The condition $b^n \equiv b \pmod{p}$ is satisfied when $n \equiv 1 \pmod{p-1}$ and the condition $p|(n+1)$ is satisfied when $n \equiv -1 \pmod{p}$. Therefore we have proven the problem for $a = 1$. □

2.3.1 Problems for the reader

2.1. (2003 Polish) Find all polynomials W with integer coefficients satisfying the following condition: for every natural number n, $2^n - 1$ is divisible by $W(n)$.

2.4 The equation $x^2 \equiv -1 \pmod p$

Theorem 2.4.1 (Wilson's). $(p-1)! \equiv -1 \pmod p$ *for all odd primes* p.

Proof. Start out with looking at a few examples. $p = 5$ gives $4! = 24 \equiv -1 \pmod 5$. However, another way to compute this is to note that

$$4! = (1 \cdot 4)(2 \cdot 3) \equiv (4)(6) \equiv (1)(4) \equiv -1 \pmod 5.$$

We test $p = 7$ which gives $6! = 720 = 7(103) - 1 \equiv -1 \pmod 7$ Also,

$$6! = (1 \cdot 6)\,(2 \cdot 4)\,(3 \cdot 5) = (6)(8)(15) \equiv (6)(1)(1) \equiv -1 \pmod 7.$$

What we are doing is we are looking at all the terms in $(p-1)!$ and finding groups of two that multiply to 1 mod p. I will now prove that this method always works.

Notice that for all $x \in \{2, 3, \cdots, p-2\}$ there exists a $y \neq x$ such that $xy \equiv 1 \pmod p$ by Theorem 2 and the fact that $x^2 \equiv 1 \pmod p \iff x \equiv \pm 1 \pmod p$. Since p is odd, we can pair the inverses off into $\frac{p-3}{2}$ pairs. Let these pairs be $(x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots (x_{(p-3)/2}, y_{(p-3)/2})$

Therefore,

$$\begin{aligned}(p-1)! &\equiv (1)(p-1)\,(x_1 y_1)\,(x_2 y_2) \cdots \left[x_{(p-3)/2} y_{(p-3)/2}\right] \pmod p \\ &\equiv (1)(p-1)\,(1)\,(1) \cdots (1) \pmod p \\ &\equiv -1 \pmod p.\end{aligned}$$

□

Theorem 2.4.2. *There exists an x with $x^2 \equiv -1 \pmod p$ if and only if $p \equiv 1 \pmod 4$.*

Proof. For the first part notice that

$$\begin{aligned}x^2 &\equiv -1 \pmod p \\ (x^2)^{\frac{p-1}{2}} &\equiv (-1)^{\frac{p-1}{2}} \pmod p \\ x^{p-1} &\equiv (-1)^{\frac{p-1}{2}} \pmod p \\ 1 &\equiv (-1)^{\frac{p-1}{2}} \pmod p,\end{aligned}$$

which happens only when $p \equiv 1 \pmod 4$.

Now proving the other way requires a bit more work. First off from Wilsons theorem $(p-1)! \equiv -1 \pmod{p}$. Therefore our equation turns into $x^2 \equiv (p-1)! \pmod{p}$. Notice that

$$\begin{aligned}(p-1)! &= [(1)(p-1)]\,[(2)(p-2)]\cdots\left[\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)\right]\\ &\equiv [(1)(-1)]\,[(2)(-2)]\left[\left(\frac{p-1}{2}\right)\left(-\frac{p-1}{2}\right)\right] \pmod{p}\\ &\equiv (-1)^{\frac{p-1}{2}}\left[\left(\frac{p-1}{2}\right)!\right]^2 \pmod{p}\\ &\equiv \left[\left(\frac{p-1}{2}\right)!\right]^2 \pmod{p}\end{aligned}$$

Therefore $x = \left(\frac{p-1}{2}\right)!$ solves the equation and we have proven the existence of x. □

> **Example 2.4.1.** *Prove that there are no positive integers x, k and $n \geq 2$ such that $x^2 + 1 = k(2^n - 1)$.*

Solution. Notice that the condition is equivalent to $x^2 + 1 \equiv 0 \pmod{2^n - 1}$. Since $2^n - 1 \equiv 3 \pmod{4}$ it follows that there exists a prime divisor of $2^n - 1$ that is of the form $3 \pmod{4}$ (if not then $2^n - 1 \equiv 1 \pmod{4}$). Label this prime divisor to be p. Therefore we have

$$x^2 + 1 \equiv 0 \pmod{p} \rightarrow\leftarrow$$

□

> **Example 2.4.2** (Korea 1999)**.** *Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $(2^n - 1)/3$ is a divisor of $4m^2 + 1$ for some integer m.*

Solution. Since $2^n - 1 \equiv 0 \pmod{3}$, we must have $n = 2k$. Therefore, our desired equation is $\frac{4^k-1}{3} | 4m^2 + 1$. I claim that $k = 2^m$ for $m \geq 0$ is a solution to this. To prove this requires using difference of squares and then the Chinese remainder theorem:

$$\begin{aligned}\frac{4^k - 1}{3} &= \frac{4^{2^m} - 1}{3} = \frac{(4^{2^{m-1}} + 1)(4^{2^{m-2}} + 1)\cdots(4+1)(4-1)}{3}\\ &= (4^{2^{m-1}} + 1)(4^{2^{m-2}} + 1)\cdots(4+1).\end{aligned}$$

We must have $4m^2+1 \equiv 0 \pmod{(4^{2^{m-1}}+1)(4^{2^{m-2}}+1)\cdots(4+1)}$. To use the Chinese Remainder Theorem to seperate this into different parts we must have all numbers in the product to be relatively prime. To prove this we notice that $4^{2^a}+1 = 2^{2^{a+1}}+1$ and then use the following lemma.

Lemma. Define $F_m = 2^{2^m}+1$ to be a Fermat number. Prove that all Fermat numbers are pairwise relatively prime.

Proof. ([12])

Start out with noting that $F_m = F_{m-1}F_{m-2}\cdots F_0 + 2$. To prove this we use induction. Notice that for $m=1$, we have $F_1 = F_0+2$ since $F_1 = 2^{2^1}+1 = 5$ and $F_0 = 2^{2^0}+1 = 3$. Assume this holds for $m=k$ and I prove it holds for $m=k+1$.

$$\begin{aligned}
F_{m+1} &= F_mF_{m-1}\cdots F_0+2\\
\iff F_{m+1} &= F_m(F_m-2)+2 \text{ from inductive hypothesis}\\
\iff F_{m+1} &= F_m^2-2F_m+2\\
\iff F_{m+1} &= (F_m-1)^2+1\\
\iff F_{m+1} &= (2^{2^m})^2+1
\end{aligned}$$

The last step is true by definition therefore our induction is complete.

Now for $0 \le i \le m-1$,

$$\gcd(F_m, F_i) = \gcd\left[F_{m-1}F_{m-2}\cdots F_0 + 2 - F_i(F_{m-1}F_{m-2}\cdots F_{i+1}F_{i-1}\cdots F_0), F_i\right] = \gcd(2, F_i) = 1.$$

When we range m over all possible non-negative integers we arrive at the conclusion that all pairs of Fermat numbers are relatively prime. $\square$

Notice that each term in the mod is the same as a Fermat number as $4^{2^a} = 2^{2^{a+1}} = F_{a+1}$ for $a \ge 0$. Hence, by Chinese Remainder Theorem the condition

$$4m^2+1 \equiv 0 \pmod{(4^{2^{m-1}}+1)(4^{2^{m-2}}+1)(\cdots)(4+1)}$$

is the same as

$$\begin{cases}
4m^2+1 \equiv 0 \pmod{4^{2^{m-1}}+1},\\
4m^2+1 \equiv 0 \pmod{4^{2^{m-2}}+1},\\
\cdots\\
4m^2+1 \equiv 0 \pmod{4+1}.
\end{cases}$$

Now, notice that $4^{2^a}+1 = 4(4^{2^a-1})+1 = 4\left(2^{2^a-1}\right)^2+1$ for $a \ge 0$. Therefore, the solution to the equation $4m^2+1 \equiv 0 \pmod{4^{2^a}+1}$ is $m \equiv 2^{2^a-1} \pmod{4^{2^a}+1}$. Therefore by Chinese Remainder Theorem there exists an m that satisfies all the above congruences and we have proved $k = 2^m$ for $m \ge 0$ works.

Assume that $k \neq 2^m$ and hence $k = p2^m$ for $m \geq 0$ and p being an odd integer that is not 1. Then

$$\frac{4^{2^m p} - 1}{3} = \frac{(4^{2^{m-1}p} + 1)(4^{2^{m-2}p} + 1) \cdots (4^p + 1)(4^p - 1)}{3}.$$

Notice that we must have

$$4m^2 + 1 \equiv 0 \left(\bmod \frac{4^{2^m * p} - 1}{3}\right) \implies 4m^2 + 1 \equiv 0 \left(\bmod \frac{4^p - 1}{3}\right)$$

Notice that $\frac{4^p - 1}{3} = \frac{(2^p - 1)(2^p + 1)}{3}$. However, since p is odd we have $3 \mid 2^p + 1$ so

$$4m^2 + 1 \equiv 0 \left(\bmod \frac{4^p - 1}{3}\right) \implies 4m^2 + 1 \equiv 0 \left(\bmod 2^p - 1\right)$$

Since $p > 1$ we clearly have $2^p - 1 \equiv 3 \pmod 4$. Assume that all prime divisors of $2^p - 1$ are of the form $1 \pmod 4$. However, this would imply that $2^p - 1 \equiv 1 \pmod 4$ a contradiction therefore there exists at least one prime divisor of $2^p - 1$ that is of the form $3 \pmod 4$. Label this prime divisor to be z.

$$(2m)^2 \equiv -1 \left(\bmod \frac{4^p - 1}{3}\right) \equiv -1 \pmod z$$

but by Theorem 3 this is a contradiction. Henceforth the answer is $k = 2^m$ for $m \geq 0$; therefore $n = 2^r$ for $r \geq 1$. □

2.5 Order

Definition 2.5.1. The *order* of a mod m (with a and m relatively prime) is the smallest positive integer x such that $a^x \equiv 1 \pmod m$. We write this as $x = \text{ord}_m a$ or sometimes shorthanded to $o_m a$.

Example. The order of 2 mod 9 is 6 because $2^1 \equiv 2 \pmod 9, 2^2 \equiv 4 \pmod 9, 2^3 \equiv 8 \pmod 9, 2^4 \equiv 7 \pmod 9, 2^5 \equiv 5 \pmod 9, 2^6 \equiv 1 \pmod 9$.

Example. Prove that the order of a mod m (with a and m relatively prime) is less than or equal to $\phi(m)$.

Proof. By Euler's Totient theorem we have

$$a^{\phi(m)} \equiv 1 \pmod m$$

Since order is the smallest x such that $a^x \equiv 1 \pmod m$ it follows that $x \leq \phi(m)$. □

The following theorem is incredibly important and we will use it a lot throughout the document.

Theorem 2.5.1. *For relatively prime positive integers a and m prove that $a^n \equiv 1 \pmod{m}$ if and only if*

$$ord_m a \mid n$$

Proof. If $\text{ord}_m a \mid n$ then we have $n = \text{ord}_m a q$ for some q. Therefore

$$a^n \equiv \left(a^{\text{ord}_m a}\right)^q \equiv 1 \pmod{m}$$

Now to prove the other direction. By the division algorithm we can write

$$n = (\text{ord}_m a)\, q + r \qquad\qquad 0 \le r < \text{ord}_a m, q \in \mathbb{Z}$$

We now notice that

$$\begin{aligned} a^{(\text{ord}_m a)q+r} = a^n &\equiv 1 \pmod{m} \\ \implies a^{(\text{ord}_m a)q} a^r &\equiv 1 \pmod{m} \\ \implies a^r &\equiv 1 \pmod{m} \end{aligned}$$

However $r < \text{ord}_m a$ contradicting the minimality of $\text{ord}_m a$ if $r \neq 0$. Therefore $r = 0$ and we are done. □

Corollary 2.5.1. *For relatively prime positive integers a and m*

$$ord_m a \mid \phi(m)$$

Proof. Set $n = \phi(m)$ and use Euler's Totient theorem. □

Example 2.5.1. *For positive integers $a > 1$ and n find $ord_{a^n-1}(a)$*

Solution. We check that the order exists by noting $\gcd(a^n - 1, a) = 1$. Notice that

$$\text{ord}_{a^n-1}(a) = x \iff a^x - 1 \equiv 0 \pmod{a^n - 1}$$

For $1 \le x < n$ we have $a^x - 1 < a^n - 1$ therefore $a^x - 1 \equiv 0 \pmod{a^n - 1}$ is impossible. For $x = n$ we have $a^n - 1 \equiv 0 \pmod{a^n - 1}$. Therefore $\text{ord}_{a^n-1}(a) = \boxed{n}$. □

Example 2.5.2 (AIME 2001). *How many positive integer multiples of 1001 can be expressed in the form $10^j - 10^i$, where i and j are integers and $0 \le i < j \le 99$?*

Solution. We must have

$$\begin{gathered} 1001 \mid 10^i \left(10^{j-i} - 1\right) \\ \implies 1001 \mid 10^{j-i} - 1 \end{gathered}$$

Notice that $\operatorname{ord}_{1001}(10) = 6$ since $10^3 \equiv -1 \pmod{1001}$ and $10^1, 10^2, 10^4, 10^5 \not\equiv 1 \pmod{1001}$. By theorem 8 we must now have $6 \mid j - i$.

We now relate this problem into a counting problem. In the case that $j \equiv i \equiv 0 \pmod 6$ there are 17 values between 0 and 99 inclusive that satisfy this. We notice that if we choice two different values out of these 17 values, then one must be j and the other must be i since $i < j$. Therefore there are $\binom{17}{2}$ solutions in this case.

Simiarly, when $j \equiv i \equiv 1 \pmod 6, j \equiv i \equiv 2 \pmod 6, j \equiv i \equiv 3 \pmod 6$ we get $\binom{17}{2}$ solutions. When $j \equiv i \equiv 4 \pmod 6$ and $j \equiv i \equiv 5 \pmod 6$ we get only 16 values between 0 and 99 henceforth we get $\binom{16}{2}$ solutions.

Our answer is $4\binom{17}{2} + 2\binom{16}{2} = \boxed{784}$. □

Example 2.5.3. *Prove that if p is prime, then every prime divisor of $2^p - 1$ is greater than p.*

Solution. Let q be a prime divisor of $2^p - 1$. Therefore we have $\operatorname{ord}_q(2) \mid p$ by Theorem 8. Since p is prime we have

$$\operatorname{ord}_q(2) \in \{1, p\}$$

However, $\operatorname{ord}_q(2) = 1$ implies that $2 \equiv 1 \pmod q$ absurd. Therefore $\operatorname{ord}_q(2) = p$.

We also have

$$\operatorname{ord}_q(2) \mid \phi(q) \implies p \mid q - 1$$

Therefore $q \ge p + 1$ and hence $q > p$ and we are done. □

Example 2.5.4. *Let p be an odd prime, and let q and r be primes such that p divides $q^r + 1$. Prove that either $2r \mid p - 1$ or $p \mid q^2 - 1$.*

Solution. Because p is odd and $p \mid q^r+1$ we have $p \nmid q^r-1$. Also we have $p \mid q^{2r}-1$. Henceforth by Theorem 8

$$\operatorname{ord}_p(q) \mid 2r$$

However since $\operatorname{ord}_p(q) \neq r$ we have $\operatorname{ord}_p(q) \in \{1, 2, 2r\}$.

When $\operatorname{ord}_p(q) = 2r$ we have $\operatorname{ord}_p(q) \mid \phi(p) = p-1$ we have $2r \mid p-1$.

When $\operatorname{ord}_p(q) \in \{1, 2\}$ in the first case we get $p \mid q-1$ and in the second we get $p \mid q^2-1$. In conclusion we get $p \mid q^2-1$.

Therefore $2r \mid p-1$ *or* $p \mid q^2-1$ □

Example 2.5.5. *Let $a > 1$ and n be given positive integers. If p is an odd prime divisor of $a^{2^n}+1$, prove that $p-1$ is divisible by 2^{n+1}.*

Solution. Since p is odd we have $p \nmid a^{2^n}-1$. We also have $p \mid \left(a^{2^n}-1\right)\left(a^{2^n}+1\right) = a^{2^{n+1}}-1$. Therefore

$$\operatorname{ord}_p(a) \mid 2^{n+1}$$

However $\operatorname{ord}_p(a) \nmid 2^n$ (since if it did then we would have $p \mid a^{2^n}-1$) hence

$$\operatorname{ord}_p(a) = 2^{n+1}$$

We know that $\operatorname{ord}_p(a) \mid p-1$ hence $2^{n+1} \mid p-1$. □

Example 2.5.6 (Classical)**.** *Let n be an integer with $n \geq 2$. Prove that n doesn't divide 2^n-1.*

Solution. Let p be the smallest prime divisor of n. We have

$$p \mid n \mid 2^n-1$$

By theorem 8 it follows that $\operatorname{ord}_p(2) \mid n$. However notice that $\operatorname{ord}_p(2) \leq \phi(p) < p$ contradicting p being the smallest prime divisor of n. □

Example 2.5.7. *Prove that for all positive integers $a > 1$ and n we have $n \mid \phi(a^n-1)$.*

Solution. Since $\gcd(a^n-1, a) = 1$ we consider $\operatorname{ord}_{a^n-1}(a)$. From above, we have $\operatorname{ord}_{a^n-1}(a) = n$

Now by theorem 8 we have $\operatorname{ord}_{a^n-1}(a) \mid \phi(a^n-1)$ hence $n \mid \phi(a^n-1)$ as desired. □

Motivation. The motivation to work mod $a^n - 1$ stems from the fact that we want a value to divide $\phi(a^n - 1)$ so hence we will likely use order mod $a^n - 1$. After a bit of thought we think to look at $\text{ord}_a(a^n - 1)$. □

Example 2.5.8. *If a and b are positive integers relatively prime to m with $a^x \equiv b^x \pmod{m}$ and $a^y \equiv b^y \pmod{m}$ prove that*

$$a^{\gcd(x,y)} \equiv b^{\gcd(x,y)} \pmod{m}.$$

Solution. We have that

$$\begin{aligned} a^x &\equiv b^x \pmod{m} \\ b^x\left[\left(a \cdot b^{-1}\right)^x - 1\right] &\equiv 0 \pmod{m} \\ \left(a \cdot b^{-1}\right)^x &\equiv 1 \pmod{m} \end{aligned}$$

Set $z \equiv a \cdot b^{-1} \pmod{m}$. From $z^x \equiv 1 \pmod{m}$ we have $\text{ord}_m(z) \mid x$. Simiarly we have $\text{ord}_m(z) \mid y$ therefore

$$\text{ord}_m(z) \mid \gcd(x, y)$$

From this we arrive at

$$\begin{aligned} z^{\gcd(x,y)} &\equiv 1 \pmod{m} \\ \left(a \cdot b^{-1}\right)^{\gcd(x,y)} &\equiv 1 \pmod{m} \\ a^{\gcd(x,y)} &\equiv b^{\gcd(x,y)} \pmod{m} \end{aligned}$$

□

Example 2.5.9. *Let a and b be relatively prime integers. Prove that any odd divisor of $a^{2^n} + b^{2^n}$ is of the form $2^{n+1}m + 1$.*

Solution. It suffices to prove that all prime divisors of $a^{2^n} + b^{2^n}$ are $1 \pmod{2^{n+1}}$. If we could do this, by multiplying the prime divisors together we get that all divisors are of the form $1 \pmod{2^{n+1}}$. We let q be an odd divisor of $a^{2^n} + b^{2^n}$. Since $\gcd(a, b) = 1$ it follows that $\gcd(q, a) = 1$ and $\gcd(q, b) = 1$.

$$\begin{aligned} q &\mid a^{2^n} + b^{2^n} \\ q &\mid a^{2^n}\left[1 + \left(b \cdot a^{-1}\right)^{2^n}\right] \\ q &\mid \left(b \cdot a^{-1}\right)^{2^n} + 1 \\ q &\mid \left(b \cdot a^{-1}\right)^{2^{n+1}} - 1 \end{aligned}$$

Let $z \equiv b \cdot a^{-1} \pmod{q}$. Therefore $\operatorname{ord}_q(z) \mid 2^{n+1}$. However since q is odd we have $q \nmid z^{2^n} - 1$ therefore

$$\operatorname{ord}_q(z) = 2^{n+1} \mid q - 1$$

$\square$

Example 2.5.10 (Bulgaria 1996)**.** *Find all pairs of prime p, q such that $pq \mid (5^p - 2^p)(5^q - 2^q)$.*

Solution. We must either have $p \mid 5^p - 2^p$ or $p \mid 5^q - 2^q$. Assume $p \mid 5^p - 2^p$. By Fermat's Little Theorem we arrive at

$$5^p - 2^p \equiv 3 \equiv 0 \pmod{p} \implies p = 3$$

We must either have $q \mid (5^3 - 2^3) = 117$ or $q \mid (5^q - 2^q)$. By the same method, the second one produces $q = 3$ while the first renders $q = 13$. Using symettry we arrive at the solutions $(p, q) = (3, 3), (3, 13), (13, 3)$. Assume that $p, q \neq 3$ now.

We must have $p \mid 5^q - 2^q$ and $q \mid 5^p - 2^p$ then.

$$\begin{aligned} 5^q - 2^q &\equiv 0 \pmod{p} \\ 2^q\left[\left(5 \cdot 2^{-1}\right)^q - 1\right] &\equiv 0 \pmod{p} \\ \left(5 \cdot 2^{-1}\right)^q - 1 &\equiv 0 \pmod{p} \end{aligned}$$

Using this same logic we arrive at

$$\left(5 \cdot 2^{-1}\right)^p - 1 \equiv 0 \pmod{q}$$

Let $a \equiv 5 \cdot 2^{-1} \pmod{pq}$. We therefore have

$$\begin{cases} \operatorname{ord}_p(a) \mid q \\ \operatorname{ord}_q(a) \mid p \end{cases} \implies \begin{cases} \operatorname{ord}_p(a) \in \{1, q\} \mid \phi(p) \\ \operatorname{ord}_q(a) \in \{1, p\} \mid \phi(q) \end{cases}$$

If $\text{ord}_p(a) = q$ and $\text{ord}_q(a) = p$ then by Theorem 8 we would have

$$\begin{cases} q \mid p-1 \\ p \mid q-1 \end{cases} \implies \begin{cases} p \geq q+1 \\ q \geq p+1 \end{cases}$$

absurd. Therefore assume WLOG that $\text{ord}_p(a) = 1$ therefore $a - 1 \equiv 0 \pmod p$. Since $a \equiv 5 \cdot 2^{-1} \pmod p$ we have

$$2(a-1) \equiv (2)(5)\left(2^{-1}\right) - 2 \equiv 3 \equiv 0 \pmod p$$

contradicting $p, q \neq 3$.

The solutions are hence $(p, q) = \boxed{(3,3),(3,13),(13,3)}$. □

Example 2.5.11 (USA TST 2003)**.** *Find all ordered prime triples* (p, q, r) *such that* $p \mid q^r + 1$*,* $q \mid r^p + 1$*, and* $r \mid p^q + 1$*.*

Solution. We split this up into two cases.

Case 1: Assume that $p, q, r \neq 2$. We therefore have $p \nmid q^r - 1$ and similarly. Therefore we have

$$\begin{cases} p \mid q^{2r} - 1 \\ q \mid r^{2p} - 1 \\ r \mid p^{2q} - 1 \end{cases} \implies \begin{cases} \text{ord}_p(q) \in \{1, 2, 2r\} \mid p - 1 \\ \text{ord}_q(r) \in \{1, 2, 2p\} \mid q - 1 \\ \text{ord}_r(p) \in \{1, 2, 2q\} \mid r - 1 \end{cases}$$

because $\text{ord}_p(q) \mid 2r$ and $\text{ord}_p(q) \neq r$ and similarly.

Assume that $\text{ord}_p(q), \text{ord}_q(r), \text{ord}_r(p) \in \{1, 2\}$. $\text{ord}_p(q) = 1$ implies $q \equiv 1 \pmod p$ and $\text{ord}_p(q) = 2$ implies $q^2 \equiv 1 \pmod p \implies q \equiv -1 \pmod p$. Therefore we arrive at

$$\begin{cases} q \equiv \pm 1 \pmod p \\ r \equiv \pm 1 \pmod q \\ p \equiv \pm 1 \pmod r \end{cases} \implies \begin{cases} q \pm 1 \geq 2p \\ r \pm 1 \geq 2q \\ p \pm 1 \geq 2r \end{cases}$$

since $q \pm 1 \neq p$ from them all being odd. This inequality set is clearly impossible.

Therefore WLOG we set $\text{ord}_p(q) = 2r$. We must have $2r \mid p - 1$ (Theorem 8) but since p and r are odd this reduces down to $r \mid p - 1$. However $r \mid p^q + 1$ but $p^q + 1 \equiv 2 \neq 0 \pmod r$ contradiction.

Case 2: We conclude there are no solutions when $p, q, r \neq 2$. Therefore WLOG let $p = 2$. From $p \mid q^r + 1$ we arrive at $q \equiv 1 \pmod 2$. Also r is odd since $r \mid 2^q + 1$. Therefore we have $r \nmid 2p^q - 1$ and $q \nmid r^2 - 1$.

$$\begin{cases} q \mid r^4 - 1 \\ r \mid 2^{2q} - 1 \end{cases} \implies \begin{cases} \text{ord}_q(r) \in \{1, 4\} \mid q - 1 \\ \text{ord}_r(2) \in \{1, 2, 2q\} \mid r - 1 \end{cases}$$

If $\text{ord}_r(2) = 1$ then we have $2 \equiv 1 \pmod r$ absurd. If $\text{ord}_r(2) = 2$ then we have $2^2 \equiv 1 \pmod r$ or $r = 3$. We must have $q \mid 3^4 - 1 = 80$. Since $q \neq 2$, we check if $q = 5$ satisfies the equation. $(p, q, r) = (2, 5, 3)$ gives us $2 \mid 5^3+1, 5 \mid 3^2+1, 3 \mid 2^5+1$ which are all true. We arrive at the solutions $(p, q, r) = (2, 5, 3), (3, 2, 5), (5, 3, 2)$.

Now we must have $\text{ord}_r(2) = 2q$. By theorem 8 we have $2q \mid r - 1$. Therefore since $q \neq 2$ we have $r \equiv 1 \pmod q$. However $q \mid r^p + 1$ but $r^p + 1 \equiv 2 \neq 0 \pmod q$.

In conclusion our solutions are $(p, q, r) = \boxed{(2, 5, 3), (3, 2, 5), (5, 3, 2)}$. □

Example 2.5.12. *Prove that for $n > 1$ we have $n \nmid 2^{n-1} + 1$.*

Proof. Let $p \mid n$. We arrive at

$$\begin{cases} p \mid 2^{p-1} - 1 \\ p \mid 2^{2n-2} - 1 \end{cases} \implies p \mid 2^{\gcd(p-1, 2n-2)} - 1$$

We must have $v_2(p - 1) \geq v_2(2n - 2)$ since $n \mid 2^{n-1} + 1$ and $p \neq 2$.. Let $n - 1 = 2^{v_2(n-1)}m$. We arrive at

$$p - 1 \equiv 0 \pmod{2^{v_2(n-1)+1}} \implies n \equiv \prod p \equiv 1 \pmod{2^{v_2(n-1)+1}}$$

However this implies $v_2(n - 1) \geq v_2(n - 1) + 1$ clearly absurd. □

Example 2.5.13. *(China 2009) Find all pairs of primes p, q such that*

$$pq \mid 5^p + 5^q$$

.

Solution. We divide this solution into two main cases.

Case 1: $p = q$

In this case we must have

$$p^2 \mid 2 \cdot 5^p \implies p = q = 5$$

Case 2: $p \neq q$. When $q = 5$ we arrive at $p \mid 5^p + 5^5$ or therefore

$$5^p + 5^5 \equiv 5 + 5^5 \equiv 0 \pmod p \implies p = 2, 5, 313$$

Hence we arrive at $(p, q) = (5, 2), (5, 313), (2, 5), (313, 5)$. Assume $p, q \neq 5$. We now have

$$\begin{aligned}
5^p + 5^q \equiv 5 + 5^q \equiv 0 \pmod{p} \implies 5^{q-1} + 1 &\equiv 0 \pmod{p} \\
1 + 5^{p-1} &\equiv 0 \pmod{q} \\
\implies 5^{2q-2} \equiv 1 \pmod{p} \text{ and } 5^{2p-2} &\equiv 1 \pmod{q} \\
\implies \operatorname{ord}_p(5) \mid \gcd(2q-2, p-1) \text{ and } \operatorname{ord}_q(5) &\mid \gcd(2p-2, q-1)
\end{aligned}$$

So long as $p, q \neq 2$ we arrive at

$$v_2\left(\operatorname{ord}_p 5\right) = v_2(2q-2) \leq v_2(p-1) \text{ and } v_2\left(\operatorname{ord}_q 5\right) = v_2(2p-2) \leq v_2(q-1)$$

Since $5^{q-1} \not\equiv 1 \pmod{p}$ and $5^{p-1} \not\equiv 1 \pmod{q}$. The two equations above are clearly a contradiction. Therefore p or q is 2. Let $p = 2$ for convenience. We then must have

$$5^2 + 5^q \equiv 5^2 + 5 \equiv 0 \pmod{q} \implies q = 2, 3, 5$$

We however know that $p \neq q$ therefore we rule out that solution. The solutions are hence $(p, q) = (2, 3), (2, 5), (3, 2), (5, 2)$.

In conclusion the solutions are

$$(p, q) = \boxed{(2, 3), (2, 5), (3, 2), (5, 2), (5, 5), (5, 313), (313, 5)}.$$

$\square$

3

p-adic Valuation

3.1 Definition and Basic Theorems

Important: *Unless otherwise stated p is assumed to be a prime.*

Definition 3.1.1. We define the **p-adic valuation** of m to be the highest power of p that divides m. The notation for this is $v_p(m)$.

Example. Since $20 = 2^2 \cdot 5$ we have $v_2(20) = 2$ and $v_5(20) = 1$. Since $360 = 2^3 \cdot 3^2 \cdot 5^1$ we have $v_2(120) = 3, v_3(360) = 2, v_5(360) = 1$. m can be a fraction, in this case we have $v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$.

Theorem 3.1.1. $v_p(ab) = v_p(a) + v_p(b)$.

Proof. Set $v_p(a) = e_1$ and $v_p(b) = e_2$. Therefore $a = p^{e_1}a_1$ and $b = p^{e_2}b_1$ where a_1 and b_1 are relatively prime to p. We then get

$$ab = p^{e_1+e_2}a_1b_1 \implies v_p(ab) = e_1 + e_2 = v_p(a) + v_p(b)$$

$\square$

Theorem 3.1.2. *If $v_p(a) > v_p(b)$ then $v_p(a+b) = v_p(b)$.*

Proof. Again write $v_p(a) = e_1$ and $v_p(b) = e_2$. We therefore have $a = p^{e_1}a_1$ and $b = p^{e_2}b_1$. Notice that

$$a + b = p^{e_1}a_1 + p^{e_2}b_1 = (p^{e_2})\left(p^{e_1-e_2}a_1 + b_2\right)$$

Since $e_1 \geq e_2 + 1$ we have $p^{e_1-e_2}a_1 + b_2 \equiv b_2 \neq 0 \pmod{p}$ therefore $v_p(a+b) = e_2 = b$ as desired. $\square$

At this point the reader is likely pondering "these seem interesting but I do not see use for them". Hopefully the next example proves otherwise (we've split it into two parts).

Example 3.1.1. *Prove that* $\sum_{i=1}^{n} \frac{1}{i}$ *is not an integer for* $n \geq 2$.

Solution. The key idea for the problems is to find a prime that divides into the denominator more than in the numerator.

Notice that

$$\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} \frac{\frac{n!}{i}}{n!}$$

We consider $v_2\left(\sum_{i=1}^{n} \frac{n!}{i}\right)$. From 3.1.2 we get

$$v_2\left(\frac{n!}{2i-1} + \frac{n!}{2i}\right) = v_2\left(\frac{n!}{2i}\right)$$

We then get $v_2\left(\frac{n!}{4i-2} + \frac{n!}{4i}\right) = v_2\left(\frac{n!}{4i}\right)$ and repeating to sum up the factorial in this way we arrive at

$$v_2\left(\sum_{i=1}^{n} \frac{n!}{i}\right) = v_2\left(\frac{n!}{2^{\lfloor \log_2 n \rfloor}}\right)$$

However for $\sum_{i=1}^{n} \left(\frac{1}{i}\right)$ to be an integer we need

$$\begin{aligned} v_2\left(\sum_{i=1}^{n} \frac{n!}{i}\right) &\geq v_2(n!) \\ v_2\left(\frac{n!}{2^{\lfloor \log_2 n \rfloor}}\right) &\geq v_2(n!) \\ 0 &\geq \lfloor \log_2 n \rfloor, \end{aligned}$$

which is a contradiction since $n \geq 2$. $\square$

Example 3.1.2. *Prove that* $\displaystyle\sum_{i=0}^{n} \frac{1}{2i+1}$ *is not an integer for* $n \geq 1$*.*

Solution.

Definition 3.1.2. We define $(2i+1)!!$ to be the product of all odd numbers less than or equal to $2i+1$. Therefore $(2i+1)!! = (2i+1)(2i-1)\cdots 3\cdot 1$. For example $(5)!! = 5 \cdot 3 \cdot 1 = 15$.

Similarly to what was done in the previous problem, we can rewrite the summation as

$$\sum_{i=0}^{n} \frac{1}{2i+1} = \sum_{i=0}^{n} \frac{\frac{(2n+1)!!}{2i+1}}{(2n+1)!!}.$$

Notice that

$$v_3\left(\frac{(2n+1)!!}{3i-2} + \frac{(2n+1)!!}{3i} + \frac{(2n+1)!!}{3i+2}\right) = v_3\left(\frac{(2n+1)!!}{3i}\right).$$

Since $v_3\left(\frac{(2n+1)!!}{3i-2} + \frac{(2n+1)!!}{3i+2}\right) > v_3\left(\frac{(2n+1)!!}{3i}\right)$. Repeating to sum all terms up in groups of three as this, we arrive at

$$v_3\left(\sum_{i=0}^{n} \frac{(2n+1)!!}{2i+1}\right) \geq v_3\left(\frac{(2n+1)!!}{3^{\lfloor \log_3(2n+1) \rfloor}}\right).$$

However we must have

$$\begin{aligned} v_3\left(\frac{(2n+1)!!}{3^{\lfloor \log_3(2n+1) \rfloor}}\right) &\geq v_3\left[(2n+1)!!\right] \\ 0 &\geq \lfloor \log_3(2n+1) \rfloor \end{aligned}$$

which is a contradiction since $n \geq 1$.

□

Example 3.1.3 (ISL 2007 N2)**.** *Let* $b, n > 1$ *be integers. For all* $k > 1$*, there exists an integer* a_k *so that* $k \mid (b - a_k^n)$*. Prove that* $b = m^n$ *for some integer* m*.*

Solution. Assume to the contrary and that there exists a prime p that divides into b such that $v_p(b) \not\equiv 0 \pmod{n}$. Therefore we set

$$b = p^{e_1 n + f_1} b_1, \qquad 1 \leq f_1 \leq n-1$$

where $\gcd(b_1, p) = 1$ and p is a prime. Now let $k = p^{n(e_1+1)}$. We must have $v_p(b - a_k^n) \geq v_p(k) = n(e_1+1)$.

- If $v_p(a_k) \le e_1$ then we have

$$v_p(b) > v_p(a_k^n) \implies v_p(b - a_k^n) = v_p(a_k^n) \le ne_1$$

contradiction!

- If $v_p(a_k) \ge e_1 + 1$ then we have

$$v_p(a_k^n) > v_p(b) \implies v_p(b - a_k^n) = v_p(b) < n(e_1 + 1)$$

contradiction!

Both cases lead to a contradiction, therefore $f_1 \equiv 0 \pmod{n}$ and we have $b = m^n$. □

3.2 p-adic Valuation of Factorials

Factorials are special cases that really lend themselves to p-adic valuations, as the following examples show.

Example. Find $v_3(17!)$.

Solution. We consider the set $\{1, 2, \cdots, 17\}$. We count 1 power of 3 for each element with $v_p(x) = 1$ and we count two powers of 3 for each element with $v_p(x) = 2$.

- We begin by counting how many numbers have at least one power of 3 which is simply $\lfloor \frac{17}{3} \rfloor$.
- We have already accounted for the numbers with two powers of 3, however they must be counted twice so we add $\lfloor \frac{17}{9} \rfloor$ to account for them the second time.

Our answer is hence $\lfloor \frac{17}{3} \rfloor + \lfloor \frac{17}{9} \rfloor = \boxed{6}$. □

Theorem 3.2.1 (Legendre)**.** *For all positive integers n and positive primes p, we have*

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor .$$

Outline. We consider the power of p that divides into $n!$.

- The number of occurences that the factor p^1 occurs in the set $\{1, 2, \cdots, n\}$. We notice that it occurs $\lfloor \frac{n}{p} \rfloor$ times.

- The number of occurences that the factor p^2 occur in the set $\{1, 2, \cdots, n\}$ is going to be $\lfloor \frac{n}{p^2} \rfloor$. We have added these once already when we added $\lfloor \frac{n}{p} \rfloor$, but since we need this to be added 2 times (since it is p^2), we add $\lfloor \frac{n}{p^2} \rfloor$ once.

Repeating this process as many times as necessary gives the desired $v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$. □

Theorem 3.2.2 (Legendre)**.** *For all positive integers n and positive primes p, we have*

$$v_p(n!) = \frac{n - s_p(n)}{p - 1}$$

where $s_p(n)$ denotes the sum of the digits of n in base p.

Proof. In base n write

$$n = \sum_{i=0}^{k} \left(a_i \cdot p^i\right) \qquad p - 1 \geq a_k \geq 1 \text{ and } p - 1 \geq a_i \geq 0 \text{ for } 0 \leq i \leq k - 1$$

From 3.2.1 we arrive at

$$\begin{aligned} v_p\left(n!\right) &= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{j=1}^{k} a_j \left(p^{j-1} + p^{j-2} + \cdots + 1\right) \\ &= \sum_{j=1}^{k} a_j \left(\frac{p^j - 1}{p - 1}\right) \end{aligned}$$

where our steps were motivated by examining $\sum_{i=1}^{\infty} \left\lfloor \frac{a_j p^j}{p^i} \right\rfloor$. [1]

[1] To make this more rigorous we would use the double summation notation. I avoided this notation to make the proof more easier to follow.

Now we evaluate $\dfrac{n-s_p(n)}{p-1}$. We notice that $s_p(n)=\displaystyle\sum_{i=0}^{k} a_i$, which implies that

$$\begin{aligned}\frac{n-s_p(n)}{p-1} &= \frac{1}{p-1}\left[\sum_{i=0}^{k}\left(a_i\cdot p^i\right)-\sum_{i=0}^{k}\left(a_i\right)\right]\\ &= \frac{1}{p-1}\sum_{i=0}^{k}\left[a_i\left(p^i-1\right)\right].\end{aligned}$$

Therefore $v_p\left(n!\right)=\dfrac{n-s_p(n)}{p-1}$ as desired. □

Example 3.2.1 (Canada)**.** *Find all positive integers n such that $2^{n-1}\mid n!$.*

Solution. From 3.2.2 we have

$$v_2\left(n!\right)=n-s_2\left(n\right)\geq n-1 \implies 1\geq s_2\left(n\right)$$

This happens only when n is a power of 2. □

Example 3.2.2. *Find all positive integers n such that $n\mid(n-1)!$.*

Solution. Set $n=p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ be the prime factorization of n. If $k\geq 2$ then write $n=\left(p_1^{e_1}\right)\left(p_2^{e_2}\cdots p_k^{e_k}\right)$. From Chinese Remainder Theorem we must prove that

$$\left(p_1^{e_1}\right)\mid(n-1)!\text{ and }\left(p_2^{e_2}\cdots p_k^{e_k}\right)\mid(n-1)!$$

However since $n-1>p_1^{e_1}$ and $n-1>p_2^{e_2}\cdots p_k^{e_k}$ this is true.

Therefore we now consider $k=1$ or $n=p_1^{e_1}$. Using Legendre's formula we have

$$v_{p_1}\left(p_1^{e_1}-1\right)!=\sum_{i=1}^{\infty}\left\lfloor\frac{p_1^{e_1}-1}{p_1^i}\right\rfloor\geq e_1.$$

For $e_1=1$ this statement is obviously false (which correlates to n being prime). For $e_1=2$ we must have

$$\left\lfloor\frac{p_1^2-1}{p_1}\right\rfloor\geq e_1 \implies p_1-1\geq e_1$$

Therefore since $e_1 = 2$ we have $p_1 = 2$ giving the only counterexample (which correlates to $n = 4$). Now for $e_1 \geq 3$ we have

$$\sum_{i=1}^{\infty} \left\lfloor \frac{p_1^{e_1} - 1}{p_1^i} \right\rfloor \geq p_1^{e_1 - 1} - 1 \geq 2^{e_1 - 1} - 1 \geq e_1.$$

Therefore the anwer is $\boxed{\text{all composite } n \text{ not equal to } 4}$.

□

Example 3.2.3. *Prove that for any positive integer n, the quantity $\frac{1}{n+1}\binom{2n}{n}$ is an integer.* ***Do not use binomial identities.***

Solution. For $n + 1 = p$ prime the problem claim is true by looking at the sets $\{n + 1, n + 2, \cdots, 2n\}$ and $\{1, 2, \cdots, n\}$ and noticing that factors of p only occur in the first set. Otherwise let p be a prime divisor of $n + 1$ less than $n + 1$. We must prove that

$$v_p \left[\binom{2n}{n} \right] \geq v_p (n + 1)$$

Let $v_p (n + 1) = k$. Therefore write $n + 1 = p^k n_1$ where $\gcd(n_1, p) = 1$. By Legendre's we have

$$v_p \left[\binom{2n}{n} \right] = \sum_{i=1}^{\infty} \left\lfloor \frac{2n}{p^i} \right\rfloor - 2 \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

Lemma. When $n + 1 = p^k n_1$ and $k \geq m$ we have

$$\left\lfloor \frac{2n}{p^m} \right\rfloor - 2 \left\lfloor \frac{n}{p^m} \right\rfloor = 1.$$

Proof. We notice that $2n = 2n_1 p^k - 2$. Therefore for $p \neq 2$ we have

$$\left\lfloor \frac{2n}{p^m} \right\rfloor = 2n_1 p^{k-m} - 1 \quad \text{and} \quad 2 \left\lfloor \frac{n}{p^m} \right\rfloor = 2 \left(p^{k-m} n_1 - 1 \right)$$

For $p = 2$, we arrive at the same formula unless $m = 1$, which then gives

$$\left\lfloor \frac{2n}{2} \right\rfloor = 2^k n_1 - 1 \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor = 2^{k-1} n_1 - 1.$$

The calculations in both cases are left to the reader to verify (simple exercise). □

Therefore using our lemma we have

$$v_p\left[\binom{2n}{n}\right] \geq k,$$

which is what we wanted. $\square$

Example 3.2.4 (Putnam 2003). *Show that for each positive integer n,*

$$n! = \prod_{i=1}^{n} lcm \left\{1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right\}$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

Solution. If we have $v_p(a) = v_p(b)$ for all primes p then in conclusion we would have $a = b$ since the prime factorizations would be the same. Assume that for this case $p \leq n$ because otherwise it is clear that $v_p(n!) = v_p\left(\prod_{i=1}^{n} \text{lcm } \left\{1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right\}\right) = 0$

By theorem 7 we have $v_p(n!) = \sum_{i=1}^{\infty} \left(\frac{n}{p^i}\right)$. I claim that this is the same as

$$v_p\left(\prod_{i=1}^{n} \text{lcm } \left\{1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right\}\right).$$

- When $i \in \{1, 2, \cdots, \lfloor \frac{n}{p} \rfloor\}$ we have at least one power of p in lcm $\{1, 2, \ldots, \lfloor \frac{n}{i} \rfloor\}$. Therefore we add $\lfloor \frac{n}{p} \rfloor$.
- When $i \in \{1, 2, \cdots, \lfloor \frac{n}{p^2} \rfloor\}$ we have at least two powers of p in lcm $\{1, 2, \ldots, \lfloor \frac{n}{i} \rfloor\}$. However, since we need to count the power of p^2 a total of 2 times and it has already been counted once, we just add it once.

Repeating this process we arrive at $v_p\left(\prod_{i=1}^{n} \text{lcm } \left\{1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right\}\right) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$ as desired. $\square$

Example 3.2.5. *Prove that for all positive integers n, $n!$ divides*

$$\prod_{k=0}^{n-1}(2^n - 2^k).$$

Solution. Notice that

$$\begin{aligned}\prod_{k=0}^{n-1}(2^n - 2^k) &= \prod_{k=0}^{n-1} 2^k \left(2^{n-k} - 1\right) \\ &= 2^{1+2+\cdots+n-1} \prod_{k=1}^{n-1}(2^k - 1).\end{aligned}$$

The number of occurences of 2 in the prime factorization of $n!$ is quite obviously more in $\prod_{k=1}^{n-1}(2^n - 2^k)$ then in $n!$ using the theorem 7. Therefore we consider all odd primes p and look at how many times it divides into both expressions.

Lemma.

$$v_p\left(\prod_{k=1}^{n-1}(2^k - 1)\right) \geq \sum_{i=0}^{\infty} \left\lfloor \frac{n-1}{(p-1)p^i} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{n-1}{(p-1)p^{i-1}} \right\rfloor.$$

Outline. We consider the set $\{2^1 - 1, 2^2 - 1, \cdots, 2^{n-1} - 1\}$ and consider how many times the exponent p^i divides into each term. We consider the worst case scenario situation when the smallest value of x such that $2^x \equiv 1 \pmod{p^j}$ is $x = \phi(p^j)$ via Euler's Totient.

- When p^1 divides into members of this set, we have

  $$p|(2^k - 1) \implies k \equiv 0 \pmod{p-1}.$$

  This results in giving us a total of $\lfloor \frac{n-1}{p-1} \rfloor$ solutions.

- When p^2 divides into members of this set, we have

  $$p^2|(2^k - 1) \implies k \equiv 0 \pmod{p(p-1)}.$$

  We only account for this once using similar logic as to the proof of Legendre's theorem therefore we add this a total of $\lfloor \frac{n-1}{p(p-1)} \rfloor$ times.

Repeating this process in general gives us

$$\left\lfloor \frac{n-1}{\phi(p^j)} \right\rfloor = \left\lfloor \frac{n-1}{(p-1)p^{j-1}} \right\rfloor$$

and keeping in mind that it is a lower bound, we have proven the lemma. □

We desire to have $v_p\left(\prod_{k=1}^{n-1}(2^k-1)\right) \geq v_p(n!)$ which would in turn give us $n!$ divides $\prod_{k=1}^{n-1}(2^k-1)$. Quite obviously $n \geq p$ or else $v_p(n!) = 0$ and there is nothing to consider. Because of this,

$$\frac{n-1}{p-1} \geq \frac{n}{p} \implies \frac{n-1}{(p-1)p^{i-1}} \geq \frac{n}{p^i}.$$

Therefore, we arrive at

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n-1}{(p-1)p^{i-1}} \right\rfloor \geq \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor .$$

As a result,

$$v_p\left[\prod_{k=1}^{n-1}(2^k-1)\right] \geq v_p(n!)$$

and we are done. □

Motivation. The motivation behind the solution was looking at small cases. We take out the factors of 2 because we believe that there are more in the product we are interested in then in $n!$. For $n = 3$ we want $3 \mid (2^3-1)(2^2-1)(2-1)$. Obviously $3 \mid (2^2-1)$.

Similarly, for $n = 5$ we want $(5 \times 3) \mid (2^5-1)(2^4-1)(2^3-1)(2^2-1)(2^1-1)$. Notice that $5 \mid (2^4-1)$ and $3 \mid (2^2-1)$.

At this point the motivation to use Fermat's Little Theorem to find which terms are divisible by p hit me. From here, the idea to use Legendre's formula and Euler's Totient for p^k hit me and the rest was groundwork. □

3.2.1 Problems

3.1. (1968 IMO 6) For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty}\lfloor\frac{n+2^k}{2^{k+1}}\rfloor = \lfloor\frac{n+1}{2}\rfloor + \lfloor[\frac{n+2}{4}]\rfloor + \cdots + \lfloor[\frac{n+2^k}{2^{k+1}}]\rfloor + \cdots$$

3.2. (1975 USAMO 1) Prove that

$$\frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

is integral for all positive integral m and n.

3.3 Lifting the Exponent

Theorem 3.3.1. *For p being an odd prime relatively prime to integers a and b with $p \mid a-b$ then*

$$v_p\left(a^n-b^n\right) = v_p\left(a-b\right) + v_p\left(n\right).$$

Proof. ([?])

We use induction on $v_p\left(n\right)$.

Base case: We begin with the base case of $v_p\left(n\right)=0$. We have

$$v_p\left(a^n-b^n\right) = v_p\left(a-b\right) + v_p\left(a^{n-1}+a^{n-2}\cdot b+\cdots+b^{n-1}\right)$$

Now notice that

$$a^{n-1}+a^{n-2}\cdot b+\cdots+b^{n-1} \equiv n\cdot a^{n-1} \pmod p$$

from $a \equiv b \pmod p$ therefore since $\gcd(a,p)=1$ we have

$$v_p\left(a^n-b^n\right) = v_p\left(a-b\right)$$

as desired.

Second base case: It is not necessary to do two base cases, however it will help us down the road so we do it here. We prove that when $v_p\left(n\right)=1$ we have $v_p\left(a^n-b^n\right) = v_p\left(n\right)+v_p\left(a-b\right)$. Let $n=pn_1$ where $\gcd(n_1,p)=1$. We arrive at

$$v_p\left(a^{pn_1}-b^{pn_1}\right) = v_p\left[\left(a^p\right)^{n_1}-\left(b^p\right)^{n_1}\right] = v_p\left(a^p-b^p\right)$$

Now let $a = b + kp$ since we know $p \mid a - b$. We arrive at

$$(b+kp)^p - b^p = \binom{p}{1}(kp) + \binom{p}{2}(kp)^2 + \cdots + \binom{p}{p}(kp)^p$$

Using the fact that $p \mid \binom{p}{i}$ for all $1 \leq i \leq p-1$ we arrive at

$$v_p\left[(b+kp)^p - b^p\right] = v_p\left(\binom{p}{1}p\right) + v_p(k) = 2 + v_p(a-b) - 1$$

as desired.

Inductive hypothesis: Assume the statement holds for $v_p(n) = k$ and I prove it holds for $v_p(n) = k+1$. Set $n = p^{k+1}n_1$. Then we have

$$v_p\left[\left(a^{p^k}\right)^{pn_1} - \left(b^{p^k}\right)^{pn_1}\right] = v_p\left(a^{p^k} - b^{p^k}\right) + 1 = v_p(a-b) + k + 1$$

Via our second base case and inductive hypothesis. □

Corollary 3.3.1. *For p being an odd prime relatively prime to a and b with $p \mid a - b$ and n is a **odd** positive integer than*

$$v_p(a^n + b^n) = v_p(a+b) + v_p(n)$$

.

Theorem 3.3.2. *If $p = 2$ and n is even, and*

- $4 \mid x - y$ *then* $v_2(x^n - y^n) = v_2(x-y) + v_2(n)$
- $4 \mid x + y$ *then* $v_2(x^n - y^n) = v_2(x+y) + v_2(n)$

Proof. This is left to the reader, advising that they follow along the same lines as the above proof. □

Example 3.3.1 (AoPS)**.** *Let $p > 2013$ be a prime. Also, let a and b be positive integers such that $p|(a+b)$ but $p^2 \nmid (a+b)$. If $p^2|(a^{2013}+b^{2013})$ then find the number of positive integer $n \leq 2013$ such that $p^n|(a^{2013}+b^{2013})$*

Solution. The first condition is equivalent to $v_p(a+b) = 1$. We also must have $v_p(a^{2013} + b^{2013}) \geq 2$. However if $p \nmid a, b$ then we have

$$v_p\left(a^{2013} + b^{2013}\right) = v_p(a+b) + v_p(2013) = 1 \rightarrow\leftarrow$$

Therefore $p \mid a, b$ which in turn gives $p^{2013} \mid a^{2013} + b^{2013}$. Therefore the answer is all positive integers n less than or equal to 2013 or $\boxed{2013}$. $\square$

Example 3.3.2 (AMM). *Let a, b, c be positive integers such that $c \mid a^c - b^c$. Prove that $c \mid \frac{a^c - b^c}{a-b}$.*

Solution. Let
$$p \mid c, v_p(c) = x$$
If $p \nmid a - b$ then we obviously have

$$p^x \mid \frac{a^c - b^c}{a-b}$$

Therefore consider $p \mid a - b$. Using lifting the exponent so long as $p \neq 2$, we arrive at
$$v_p\left(\frac{a^c - b^c}{a-b}\right) = v_p(a-b) + v_p(c) - v_p(a+b) = x$$
When $p = 2$ we arrive at

$$v_2\left(\frac{a^c - b^c}{a-b}\right) = v_2(a-b) + v_2(a+b) + v_2(c) - 1 - v_2(a-b) \geq v_2(c)$$

$\square$

Example 3.3.3 (IMO 1999). *Find all pairs of positive integers (x, p) such that p is prime, $x \leq 2p$, and $x^{p-1} \mid (p-1)^x + 1$.*

Solution. Assume that $p \neq 2$ for the time being (we will see why later). We trivially have $x = 1$ always giving a solution. Now, let q be the minimal prime divisor of x. We notice that:

$$\begin{aligned} q &\mid \left[(p-1)^2\right]^x - 1 \\ \implies \text{ord}_q\left[(p-1)^2\right] &\mid \gcd(x, q-1) = 1 \\ \implies (p-1)^2 - 1 &\equiv 0 \pmod q \\ \implies (p-2)(p) &\equiv 0 \pmod q \end{aligned}$$

We know that $(p-1)^x + 1 \equiv 0 \pmod{q}$. However, if $p - 2 \equiv 0 \pmod{q}$ then we have

$$(p-1)^x + 1 \equiv 2 \neq 0 \pmod{q}$$

Since $p-1$ is odd. Therefore we must have $p \equiv 0 \pmod{q}$ or therefore $p = q$. Now, by lifting the exponent [2] we have:

$$v_p\left[(p-1)^x + 1\right] = 1 + v_p(x) \geq p - 1$$
$$x \geq p^{p-2} \geq 2p \text{ for } p > 3$$

However we have $x \leq 2p$. We now account for $p \in \{2, 3\}$.

$p = 2$ gives $x \mid 2$ or $x = 1, 2$. $p = 3$ gives $x^2 \mid 2^x + 1$ where $x \leq 6$. Therefore we have $x = 1, 3$. The solutions are hence $(x, p) = (1, p), (2, 2), (3, 3)$. $\square$

Example 3.3.4 (Bulgaria)**.** *For some positive integers n, the number $3^n - 2^n$ is a perfect power of a prime. Prove that n is a prime.*

Solution. Say $n = p_1 p_2 \cdots p_k$ where $p_1 \leq p_2 \leq \cdots \leq p_k$ and $k \geq 2$.

We have $3^{p_i} - 2^{p_i} \mid 3^n - 2^n$. Let $p \mid 3^{p_i} - 2^{p_i}$ for all p_i. We have $p \mid z^{p_i} - 1$ where $z \equiv 3 \cdot 2^{-1} \pmod{p}$ since $p \neq 2$. Therefore

$$\text{ord}_p(z) \mid p_i \implies \text{ord}_p(z) \mid \gcd(p_1, p_2, \cdots, p_i)$$

However $\gcd(p_1, p_2, \cdots, p_i) \in \{1, p_i\}$ with it being p_i when $p_1 = p_2 = \cdots = p_k$. But if $\gcd(p_1, p_2, \cdots, p_i) = 1$ then we have $p \mid z - 1$. However

$$2(z-1) \equiv 1 \equiv 0 \pmod{p} \rightarrow\leftarrow$$

Therefore $p_1 = p_2 = \cdots = p_k$. We must therefore consider $n = p^k$.

Let $3^{p^k} - 2^{p^k} = q^z$. Therefore $3^{p^{k-1}} - 2^{p^{k-1}} = q^{z_1}$. Therefore $\text{ord}_q(z) \mid p^{k-1}$ if $k \geq 2$. Let $\text{ord}_q(z) = p^m$. Therefore $3^{p^m} - 2^{p^m} \equiv 0 \pmod{q}$. Now we use lifting the exponent to give us

$$v_q\left[\left(3^{p^m}\right)^{p^{k-m}} - \left(2^{p^m}\right)^{p^{k-m}}\right] = v_q\left(3^{p^m} - 2^{p^m}\right) + v_q(p)$$

However if $q = p$ then we have $\text{ord}_p(z) \mid \gcd(p^k, p-1) = 1$ giving the same $p = 1$ contradiction. Therefore $v_q(p) = 0$. Hence we must have $v_q\left(3^{p^m} - 2^{p^m}\right) \geq z$. Therefore

$$3^{p^k} - 2^{p^k} > 3^{p^m} - 2^{p^m} \geq q^z$$

since $m \leq k - 1$. Therefore $k = 1$ implying that p is prime. $\square$

Comment. When we do Zsigmondy's theorem you will notice there is a lot easier solution to this. $\square$

[2] lifting the exponent is not the same for $p = 2$

Example 3.3.5 (IMO 1990)**.** *Find all natural n such that $\frac{2^n+1}{n^2}$ is an integer.*

Solution. Trivially $n = 1$ is a solution. Now assume $n \neq 1$ and define $\pi(n)$ to be the smallest prime divisor of n. Let $\pi(n) = p \neq 2$. Then we have:

$$\begin{aligned} p \mid 2^n + 1 \mid 2^{2n} - 1 \text{ and } p \mid 2^{p-1} - 1 \\ \implies p \mid 2^{\gcd(2n,p-1)} - 1 \end{aligned}$$

Now if $r \neq 2 \mid n$ then we can't have $r \mid p - 1$ because then $r \leq p - 1$ contradiction. Therefore $r = 2$ and since n is odd $\gcd(2n, p - 1) = 2$. Hence

$$p \mid 2^2 - 1 \implies p = 3$$

Let $v_3(n) = k$. By lifting the exponent we must have

$$v_3(2^n + 1) = 1 + k \geq v_3(n^2) = 2k \implies k = 1$$

Let $n = 3n_1$. $n_1 = 1$ is a solution ($2^3 + 1 = 3^2 = 9$). Assume $n_1 \neq 1$ and let $\pi(n_1) = q \neq 3$. By Chinese Remainder Theorem since $q \neq 3$ we have:

$$\begin{aligned} q \mid 8^{n_1} + 1 \mid 8^{2n_1} - 1 \text{ and } q \mid 8^{q-1} - 1 \\ \implies q \mid 8^{\gcd(2n_1,q-1)} - 1 = 63 \text{ so } q = 7 \end{aligned}$$

However $2^n + 1 \equiv 8^{n_1} + 1 \equiv 2 \pmod 7$ contradiction.

The solutions are henceforth $n = 1, 3$. □

Example 3.3.6 (China TST 2009)**.** *Let $a > b > 1$ be positive integers and b be an odd number, let n be a positive integer. If $b^n \mid a^n - 1$ prove that $a^b > \frac{3^n}{n}$.*

Solution. We fix b. I claim that the problem reduces down to b prime. Assume that we have shown the problem statement for b prime (which we will do later). Now let b be composite and say $q \mid b$ where q is prime. Then we would have $q^n \mid b^n \mid a^n - 1$. However, by our assumption $q^n \mid a^n - 1$ gives us $a^q > \frac{3^n}{n}$ therefore we have $a^b > a^q > \frac{3^n}{n}$. Therefore the problem reduces down to b prime. Let $b = p$ be prime.

We have $p^n \mid a^n - 1$. Since $p \mid a^n - 1$ we have $\text{ord}_p a \mid n$. Also by Fermat's Little Theorem we have $\text{ord}_p a \mid p - 1$. Let

$$\text{ord}_p a = x \leq p - 1$$

We now have $a^x \equiv 1 \pmod p$. Therefore $x \mid n$ and set $n = xn_1$.

Now we must have $p^n \mid (a^x)^{n_1} - 1$. By lifting the exponent we have

$$v_p\left[(a^x)^{n_1} - 1\right] = v_p(a^x - 1) + v_p(n_1) \geq n$$

Lemma. $\log_p n \geq v_p(n)$

Proof. Let $n = p^r s$. Therefore $v_p(n) = r$. We also have $\log_p n = r + \log_p s \geq r$. $\square$

Therefore we now have

$$
\begin{aligned}
v_p(a^x - 1) \geq n - v_p(n_1) &\geq n - \log_p n_1 \\
v_p(a^x - 1) &\geq \log_p \left(\frac{p^n}{n_1} \right) \\
a^b > a^x - 1 \geq p^{v_p(a^x-1)} \geq \frac{p^n}{n_1} = \frac{x \cdot p^n}{n} &\geq \frac{3^n}{n}
\end{aligned}
$$

We use the fact that p is odd in the last step since we have $p^n \geq 3^n$. $\square$

3.4 General Problems for the Reader

3.3. [Turkey] Let b_m be numbers of factors 2 of the number $m!$ (that is, $2^{b_m} | m!$ and $2^{b_m+1} \nmid m!$). Find the least m such that $m - b_m = 1990$.

4

Diophantine equations

4.1 Bounding

Example 4.1.1. *(Russia) Find all natural pairs of integers* (x, y) *such that* $x^3 - y^3 = xy + 61$.

Solution.

$$x^3 - y^3 = (x - y)\left(x^2 + xy + y^2\right) = xy + 61$$

Notice that $x > y$. Therefore we have to consider $x^2 + xy + y^2 \leq xy + 61$ or $x^2 + y^2 \leq 61$. Since $x > y$, we have

$$61 \geq x^2 + y^2 \geq 2y^2 \implies y \in \{1, 2, 3, 4, 5\}$$

$$\begin{cases} y = 1 & x^3 - x - 62 = 0 \\ y = 2 & x^3 - 2x - 69 \\ y = 3 & x^3 - 3x - 89 \\ y = 4 & x^3 - 4x - 125 \\ y = 5 & x^3 - 5x - 186 = 0 \end{cases}$$

Of these equations, we see the only working value for x is when $x = 6, y = 5$ so the only natural pair of solutions is $(x, y) = (6, 5)$. $\square$

4.2 The Modular Contradiction Method

Example 4.2.1. *Find all pairs of integers (x, y) that satisfy the equation*

$$x^2 - y! = 2001.$$

Solution. We consider what happens modulo 7. When $y \geq 7$, $y! \equiv 0 \pmod 7$, so $x^2 \equiv 2001 \equiv -1 \pmod 7$. Therefore if an x is to satisfy this equation, $x^2 \equiv 6 \pmod 7$. But by analyzing the table shown below, we can see that there are no solutions to that equation.

x	$x^2 \pmod 7$
0	0
1	1
2	4
3	2
4	2
5	4
6	1

Therefore $y < 7$.

Now we must check cases. Note that the smallest perfect square greater than 2001 is $2025 = 45^2$, and this (surprisingly) gives two valid solutions: $(x, y) = (45, 4)$ and $(x, y) = (-45, 4)$. This covers all cases with $y \leq 4$. If $y = 5$, then $x^2 = 2001 + 5! = 2121$, but this equation has no integer solutions since 2121 is divisible by 3 but not 9. If $y = 6$, then $x^2 = 2001 + 6! = 2721$, which once again has no solutions for the same reasons as above. Therefore our only solutions are $(x, y) = \boxed{(45, 4), (-45, 4)}$.

□

Tip 4.2.1. When dealing with factorials, it is often advantageous to take advantage of the fact that $m! \equiv 0 \pmod p$ for all positive integers $m \geq p$. By finding the right mod, we can reduce the number of cases significantly.

Example 4.2.2 (USAMTS)**.** *Prove that if m and n are natural numbers that*

$$3^m + 3^n + 1$$

cannot be a perfect square.

Solution. We look mod 8. Notice that we have

$$3^{2k} \equiv 1 \pmod 8 \text{ and } 3^{2k+1} \equiv 3 \pmod 8$$

Therefore we have $3^m \equiv \{1, 3\} \pmod 8$. Henceforth the possible values of $3^m + 3^n + 1 \bmod 8$ are $1+1+1, 3+1+1, 3+3+1$ which gives $3, 5, 7$.

Notice that when x is even we have $x^2 \equiv 0, 4 \pmod 8$. When $x = 2k+1$ we get $x^2 = 4k^2 + 4k + 1 \equiv 1 \pmod 8$ since $k^2 + k \equiv 0 \pmod 2$. Therefore the possible values of $x^2 \pmod 8$ are $0, 1, 4$. None of these match the values of $3^m + 3^n + 1 \pmod 8$ therefore we have a contradiction. □

Motivation. The motivation for looking mod 8 stems from trying mod 4 at first. Trying mod 4 eliminates the case m and n are both even (since then $3^m + 3^n + 1 \equiv 3 \pmod 4$) and the case when m and n are both odd (since then $3^m + 3^n + 1 \equiv 7 \equiv 3 \pmod 4$). Since we notice how close this gets us, we try mod 8. □

Example 4.2.3. *Prove that 19^{19} cannot be written as the sum of a perfect cube and a perfect fourth power.*

Solution. We look $\pmod{13}$. Notice that when $\gcd(x, 13) = 1$ we have

$$\left(x^3\right)^4 \equiv 1 \pmod{13}$$

hence substituting $y \equiv x^3 \pmod{13}$ we arrive at $y^4 \equiv 1 \pmod{13}$. The solutions to this equation are $y \equiv 1, 5, 8, 12 \pmod{13}$. Therefore $x^3 \equiv 0, 1, 5, 8, 12 \pmod{13}$.

Next when $\gcd(x, 13) = 1$ we have

$$\left(x^4\right)^3 \equiv 1 \pmod{13}$$

therefore substituting $x^4 \equiv y \pmod{13}$ gives us the equation $y^3 \equiv 1 \pmod{13}$. The solutions to this are $y \equiv 1, 3, 9$ henceforth $x^4 \equiv 0, 1, 3, 9 \pmod{13}$.

Lastly, notice that

$$19^{19} \equiv \left(6^{12}\right)\left(6^7\right) \equiv \left(6^2\right)^3 (6) \equiv (-3)^3 (6) \equiv 7 \pmod{13}$$

$x^3 \pmod{13}$	$y^4 \pmod{13}$
0	0
1	1
5	3
8	9
12	

It is clear that the sum of no two elements above will be $7 \pmod{13}$ therefore we are done. □

Tip 4.2.2. When a problem says to prove a diophantine equation has no solutions it usually has a modular solution. Finding the right mod is quite an important trick. Find the modulo which reduces the cases to as small as possible and this will likely be the right mod to work with. If not, try many different mods and see which ones work.

Example 4.2.4. *Find all solutions to the equation $x^5 = y^2 + 4$ in positive integers.*

Solution. We look $\pmod{11}$. The motivation behind this is noting that when $\gcd(x, 11) = 1$ we have $x^{10} \equiv 1 \pmod{11}$.

Notice that when $\gcd(x, 11) = 1$ we have

$$\left(x^5\right)^2 \equiv 1 \pmod{11}.$$

Therefore substituting $y \equiv x^5 \pmod{11}$ we arrive at $y^2 \equiv 1 \pmod{11}$ or $y \equiv \pm 1 \pmod{11}$ therefore $x^5 \equiv 0, 1, 10 \pmod{11}$.

The possible squares mod 11 are $0^2 \equiv 0 \pmod{11}, 1^2 \equiv 1 \pmod{11}, 2^2 \equiv 4 \pmod{11}, 3^2 \equiv 9 \pmod{11}, 4^2 \equiv 5 \pmod{11}, 5^2 \equiv 3 \pmod{11}$ since $x^2 \equiv (11 - x)^2 \pmod{11}$. Therefore $x^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}$.

Henceforth we have:

$x^5 \pmod{11}$	$y^2 \pmod{11}$
0	0
1	1
10	3
	4
	5
	9

Therefore $x^5 - y^2 \equiv 4 \pmod{11}$ is impossible and we have concluded that there are no solutions.

□

Example 4.2.5 (USAJMO 2013)**.** *Are there integers a, b such that $a^5 b + 3$*

and ab^5+3 are perfect cubes?

Solution. The cubic residues mod 9 are $-1, 0, 1$. Therefore we think to take mod 9. Assume that we can find a, b such that a^5b+3 and ab^5+3 are perfect cubes. Therefore we must have

$$\begin{cases} a^5b+3 \equiv -1, 0, 1 \pmod 9 \\ ab^5+3 \equiv -1, 0, 1 \pmod 9 \end{cases} \implies \begin{cases} a^5b \equiv 5, 6, 7 \pmod 9 \\ ab^5 \equiv 5, 6, 7 \pmod 9 \end{cases}$$

If $3 \mid a$ we would have $a^5b \equiv 0 \pmod 9$ so hence $\gcd(a, 3) = 1$ and similarly $\gcd(b, 3) = 1$. We notice that via Euler's Totient Theorem

$$x^6 \equiv 0, 1 \pmod 9$$

Therefore we must have a^5b and ab^5 to multiply to either 0 or 1 when reduced mod 9. However, $5 \cdot 5 \equiv 7 \pmod 9, 5 \cdot 7 \equiv 8 \pmod 9, 7 \cdot 7 \equiv 4 \pmod 9$ therefore we have arrived at a contradiction.

It is therefore impossibile to find a, b such that a^5b+3 and ab^5+3 are perfect cubes.

$\square$

Motivation. The motivation behind this solution was to limit the values on the possibilities for a^5b+3 and ab^5+3 mod 9. The other key idea was that via Euler's Totient $\phi(9) = 6$ therefore $(a^5b)(ab^5) \equiv 0, 1 \pmod 9$. $\square$

Example 4.2.6. *Find all solutions to the diophantine equation $7^x = 3^y + 4$ in positive integers (India)*

Solution.

Lemma. $\operatorname{ord}_{3^n}(7) = 3^{n-1}$

Sketch. First off notice that $\operatorname{ord}_{3^n}(11) = 3^i$ (prove it!) Via lifting the exponent we have

$$v_3\left(7^{3^i} - 1\right) = i + 1 \geq n$$

$\square$

We note that $(x, y) = (1, 1)$ is a solution and there is no solution for $y = 2$. Now assume $y \geq 3$. Therefore we have

$$7^x \equiv 4 \pmod 9 \implies x \equiv 2 \pmod 3$$

I claim that $x \equiv 8 \pmod 9$. From $\text{ord}_{27}(7) = 9$ we have

$$7^x - 4 \equiv 7^{x \pmod 9} - 4 \equiv 0 \pmod{27}$$

After testing $x = 2, 5, 8$ we arrive at $x \equiv 8 \pmod 9$.

We wish to make 7^x constant mod p. Therefore we find p such that $\text{ord}_p(7) = 9$. Since $7^9 - 1 = (7^3 - 1)(7^6 + 7^3 + 1)$ and $7^6 + 7^3 + 1 = 3 \cdot 37 \cdot 1063$ we take $p = 37$. Since $37 \nmid 7^3 - 1$ we have $\text{ord}_{37}(7) = 9$.

Take the original equation mod 7 and use $\text{ord}_7(3) = 6$ to give $y \equiv 1 \pmod 6$. Therefore y is odd so we write $y = 2m + 1$ to give us $3^y = 3^{2m+1} = 3 \cdot 9^m$. Now $\text{ord}_{37}(9) = 9$ so hence $9^m \equiv 9^{m \pmod 9} \pmod{37}$.

We have $3^y \equiv 7^x - 4 \equiv 7^7 - 4 \equiv 12 \pmod{37}$

$m \pmod 9$	$3^{2m+1} \pmod{37}$
0	3
1	27
2	21
3	4
4	36
5	28
6	30
7	11
8	25

Contradicting $3^y = 3^{2m+1} \equiv 12 \pmod{37}$.

Therefore the only solution is $(x, y) = (1, 1)$. □

Example 4.2.7. *Solve the diophantine equation in positive integers:* $2^x + 3 = 11^y$

Solution.

Lemma. $\text{ord}_{2^n} 11 = 2^{n-2}$

Sketch. We must have $\text{ord}_{2^n} 11 \mid 2^{n-1}$. From lifting the exponent $v_2\left(11^{2^i} - 1\right) = i + 2$ Therefore we must have $i + 2 \geq n \implies i \geq n - 2$. □

$(x, y) = (3, 1)$ is a solution to this equation and $x = 4$ isn't. Assume $x \geq 5$ now. Take the original equation mod 16 to give us

$$11^y \equiv 3 \pmod{16}$$
$$11^y \equiv 11^3 \pmod{16}$$
$$\implies y \equiv 3 \pmod 4$$

using the lemma.

Therefore

$$11^y - 3 \equiv 11^{y \pmod 8} - 3 \equiv 0 \pmod{32}$$

From which $y \equiv 7 \pmod 8$

We take $p = 2^4 + 1 = 17$ since $\text{ord}_{17}(2) = 8$.

We also have $\text{ord}_{17}(11) = 16$. Take the equation mod 11 to give $x \equiv 3 \pmod{10} \implies x \equiv 1 \pmod 2$. [1] Notice that $11^y \equiv 11^7, 11^{15} \equiv 3, 14 \pmod{17}$. Also

$x \pmod 8$	$2^x + 3 \pmod{17}$
1	5
3	11
5	1
7	12

Contradicting $2^x + 3 \equiv 11^y \pmod{17}$. Therefore the only solution is $(x, y) = (3, 1)$.

□

Example 4.2.8 (USAMO 2005)**.** *Prove that the system*

$$x^6 + x^3 + x^3y + y = 147^{157}$$
$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

has no solutions in integers x, y, and z.

Solution. ([16])

We wish to limit the possibilities for z^9; therefore we choose to look at the equation $\pmod{19}$. Notice that $147 \equiv 14 \pmod{19}$ and $157 \equiv 5 \pmod{19}$. By Fermat's Little Theorem we have

$$147^{157} \equiv 14^{157} \equiv 14^{13} \equiv 2 \pmod{19}^2$$
$$157^{147} \equiv 5^{147} \equiv 5^3 \equiv 11 \pmod{19}$$

Adding the two equations results in

$$\left[x^3\left(x^3 + y + 1\right) + y\right] + \left[y\left(x^3 + y + 1\right) + x^3 + z^9\right] \equiv [9] + [4] \pmod{19}$$
$$\implies \left(x^3 + y + 1\right)\left(x^3 + y\right) + \left(x^3 + y + 1\right) - 1 + z^9 \equiv 13 \pmod{19}$$
$$\implies \left(x^3 + y + 1\right)^2 + z^9 \equiv 14 \pmod{19}$$

[1] from $\text{ord}_{11}(2) = 10$

When $\gcd(z, 19) = 1$ we have $(z^9)^2 \equiv 1 \pmod{19} \implies z^9 \equiv \pm 1 \pmod{19}$. Therefore $z^9 \equiv -1, 0, 1 \pmod{19}$ or henceforth

$$(x^3 + y + 1)^2 \equiv 13, 14, 15 \pmod{19}$$

By work of calculations we find that the quadratic residues mod 19 are $\{0, 1, 4, 5, 6, 7, 9, 11, 16, 17\}$ therefore we have arrived at a contradiction. $\square$

Motivation. There isn't much motivation behind this solution. The one thing we have to notice is that z^9 is a floater and hence we likely want to limit the possibilities for it. To do so we take mod 19. $\square$

4.3 General Problems for the Reader

4.1. [Hong Kong TST 2002] Prove that if a, b, c, d are integers such that

$$(3a + 5b)(7b + 11c)(13c + 17d)(19d + 23a) = 2001^{2001}$$

then a is even.

5

Problem Solving Strategies

In this chapter, we explore three famous theorems in Number Theory, and end with some extremely challenging problems that highlight select problem solving techniques.

5.1 Chicken Mcnuggets anyone?

Mcdonalds once offered Chicken Mcnuggets in sets of 9 and 20 only. A question prompted from this is, assuming you only buy sets of 9 and 20 Chicken Mcnuggets and do not eat/add any during this process, what is the largest amount of Mcnuggets that is impossible to make? It turns out the answer is 151, which we will explore in this section.

Theorem 5.1.1 (Chicken Mcnugget Theorem)**.** *Prove that for relatively prime naturals m, n, the largest impossible sum of m, n (i.e. largest number not expressable in the form $mx + ny$ for x, y non-negative integers) is $mn - m - n$.*

Proof. (Experimenting)

First off, let's try a small case. Let $m = 7$ and $n = 5$. We then have to show that $5 \times 7 - 5 - 7 = 23$ is impossible to make, and every value above 23 is makable. Assume for sake of contradiction that $23 = 7x + 5y$. We then arrive at $x \in \{0, 1, 2, 3\}$ since when $x \geq 4$ this implies that $y < 0$.

$$\begin{cases} x = 0, 5y = 23 \\ x = 1, 5y = 16 \\ x = 2, 5y = 9 \\ x = 3, 5y = 2 \end{cases}$$

All of these result in contradictions. Next, notice that

$$\begin{cases} 24 = 5 \times 2 + 7 \times 2 \\ 25 = 5 \times 5 \\ 26 = 5 \times 1 + 7 \times 3 \\ 27 = 5 \times 4 + 7 \times 1 \\ 28 = 7 \times 4 \\ 29 = 5 \times 3 + 7 \times 2 \\ 30 = 5 \times 6 \\ 31 = 5 \times 2 + 7 \times 3 \\ 32 = 5 \times 5 + 7 \times 1 \\ \dots \end{cases}$$

We notice that 29 and 24 are exactly the same, except for that the factor of 5 is increased by 1 in 29. Similarly, the same holds for 25 to 30, and 26 to 31, and so forth. In fact, once we have $24, 25, 26, 27, 28$, the rest of the numbers above 23 are makable by just repeatedly adding 5.

OBSERVATIONS:

- WLOG let $n \geq m$. Then, if we can show that $mn - m - n$ is unmakable and $mn - m - n + 1, mn - m - n + 2, \cdots, mn - m - n + m$, are all makable, we can just add m repeatedly to make all numbers above $mn - m - n$.

- There seems to be an inequality related with $mn - m - n$.

Let's take a look at our equation above to show that $mn - m - n$ is unmakable. We arrive at $23 = 7x + 5y$. We think to check both mod 5 and mod 7 to see if this gives us anything. Mod 5 gives us

$$7x \equiv 23 \pmod 5 \implies 2x \equiv 3 \pmod 5 \implies x \equiv -1 \pmod 5$$

Therefore, we arrive at $x \geq 4$, contradicting $x \in \{0, 1, 2, 3\}$. Taking mod 7 gives us

$$5y \equiv 23 \equiv -5 \pmod 7 \implies y \equiv -1 \pmod 7$$

Therefore, we get $y \geq 6$ again contradiction. Both of these methods give one of the variables equal to -1 mod m or mod n. This gives us the inspiration to try this for the general equation.

Set

$$mx + ny = mn - m - n.$$

Taking mod m of this equation results in $ny \equiv -n \pmod{m}$. Now, since $\gcd(n, m) = 1$, we must have $y \equiv -1 \pmod{m}$. Now, we try $y = m - 1$ to see what happens. This gives us

$$mx + n(m-1) = mn - m - n \implies mx = -m$$

Contradicting x, y non-negative. Obviously, for y more than $m - 1$ the left hand side is way bigger than the right hand side, so we have now proven that $mn-m-n$ is impossible.

Let's try to construct how would we find x, y such that $24 = 7x + 5y$. Notice that taking this equation mod 7, we arrive at

$$5y \equiv 3 \pmod{7} \implies y \equiv 2 \pmod{7}$$

Therefore, set $y = 2$ and we arrive at $x = 2$ as well. Let's try to find 26 such that $26 = 7x + 5y$. Taking this equation mod 7, we arrive at

$$5y \equiv 5 \pmod{7} \implies y \equiv 1 \pmod{7}$$

Take $y = 1$ and we get $x = 3$. It seems like modulos could do the trick for us. Set

$$mn - m - n + k = mx + ny \tag{5.1}$$

Since $mn - m - n$ is a **symmetric polynomial** (meaning the variables are interchangable), we can assume **without loss or generality** that $n \geq m$. If instead $m \geq n$, then replace m by n to get $n \geq m$ (Sidenote: If $m \geq n$, we would have taken the original equation mod n.) In light of observation 1 above, we only have to consider

$$k \in \{1, 2, 3, \cdots, m\} \tag{5.2}$$

Now, what exactly do we have to prove?

- For every $k \in \{1, 2, 3, \cdots, m\}$, there exists a y such that x is an integer.
- For every $k \in \{1, 2, 3, \cdots, m\}$, there exists a y such that x is a non-negative integer.

Let's knock off the first item by using the modulo idea. Taking mod m of (1) gives us

$$n(y+1) \equiv k \pmod{m} \implies y + 1 \equiv kn^{-1} \pmod{m}$$

For any k, we get $y \equiv kn^{-1} - 1 \pmod{m}(*)$, resulting in x being an integer. We now must prove that x is a positive integer. Notice that for We know that $n-1 \pmod{m}$ exists since $\gcd(m, n) = 1$. Now, we must prove that x is a non-negative integer. This is a bit harder to do, as we have no idea how to even begin. We know that we want $mn - m - n + k - ny$ to be positive for every $k \in \{1, 2, 3, \cdots, m\}$. For $y = m - 2$, we notice that we get

$$mn - m - n + k - ny = n - m + k > 0^1$$

Similarly, for $y = m - y_0$ for $m \geq y_1 \geq 2$, we get

$$mn - m - n - +k = (y_0 - 1)n - m + k > 0$$

However, for $y = m - 1$, we have no idea where to begin. We do notice that, however, from (*),

$$y \equiv -1 \pmod{m} \implies kn^{-1} - 1 \equiv -1 \pmod{m} \implies k \equiv 0 \pmod{m}$$

Therefore for $y = m - 1$, we have $k = m$ giving

$$mn - m - n + k - ny = 0 \implies x = 0$$

Our proof is complete.

□

This was another awesome problem to do, as it had many key steps involved in it.

- We first off played around with some small cases (i.e $m = 5, n = 7$ and noticed that taking mod m and mod n helped solve the equation). We also figured that inequalities would be helpful as they worked when noting $x \in \{0, 1, 2, 3\}$.
- We used the ideas of mods to reduce down to $y \equiv kn^{-1} - 1 \pmod{m}$. We know that there will exist a value of m due to this equation, however, we must use inequalities to figure out how.
- We assume without loss or generality that $n \geq m$ to aid in our use of inequalities (we do this before the inequalities).
- We show that the equation will always be positive for every $k \in \{1, 2, 3, \cdots, m\}$.

The full rigorous proof below is included again to show what a complete proof looks like. The above method is included to show the reader what the mathematical method is like as a problem solver. The reader may prefer the following rigorous proof, but hopefully understood how they would go about finding this proof as problem solvers.

[1] using our WLOG

Proof. (Rigorous) Because the equation is symettric, WLOG assume that $n \geq m$. Assume that $mn - m - n = mx + ny$. Taking the equation mod m, we arrive at

$$ny \equiv -n \pmod{m} \implies y \equiv -1 \pmod{m}$$[2]

This implies that $y \geq m - 1$, however, this gives

$$mx + ny \geq mx + mn - m > mn - m - n$$

Contradiction. Now, I prove that

$$mn - m - n + k = mx + ny, k \in \{1, 2, 3, \cdots, m\}$$

The reason for this is that for $k = k_1 > m$, then we can repeatedly add m to the reduced value of k_1 mod m until we reach k_1. Our goal is to prove that for every k, there exists an x such that

- x is an integer.
- x is a non-negative integer.

Taking the equation mod m brings us to

$$n(y + 1) \equiv k \pmod{m} \implies y \equiv kn^{-1} - 1 \pmod{m} (1)$$

Using this value of y produces the first desired outcome. For the second, we must have $mn - m - n + k - ny \geq 0$. For $y = m - y_0$ and $m \geq y_0 \geq 2$, we get

$$mn - m - n + k - ny = (y_0 - 1)n - m + k > 0$$

For $y_0 = 1$, we have

$$y \equiv -1 \pmod{m} \implies kn^{-1} - 1 \equiv -1 \pmod{m} \implies k \equiv 0 \pmod{m}$$

Therefore, $k = m$, and we get

$$mn - m - n + k - ny = mn - m - n + m - n(m - 1) = 0 \implies x = 0$$

Therefore, we have proven our desired statement, and we are done. □

Example 5.1.1. *(IMO 1983) Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y, z are non-negative integers.*

[2] from $\gcd(m, n) = 1$

5.2 Vieta Jumping

Example 5.2.1 (Putnam 1988). *Prove that for every real number N, the equation*

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

has a solution for which x_1, x_2, x_3, x_4 are all integers larger than N.

Solution. Notice that a trivial solution of this equation is $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$. This is our generator of the other solutions, what we are about to do is find a form for another solution in terms of the other variables for x_1 to find a new value of x_1 that works. To do this, we have to isolate the variables to make a quadratic in terms of x_1. We arrive at the equation

$$x_1^2 - x_1(x_2x_3 + x_2x_4 + x_3x_4) + x_2^2 + x_3^2 + x_4^2 - x_2x_3x_4 = 0$$

Therefore, we notice that if the solutions for x_1 are $x_1 = r_1, r_2$, we have

$$r_1 + r_2 = x_2x_3 + x_2x_4 + x_3x_4$$

by Vietas formula. Assume that $(x_1, x_2, x_3, x_4) = (r_1, s_1, t_1, u_1)$ is a solution of the original equation with WLOG $u_1 \geq t_1 \geq s_1 \geq r_1$. Then by the above relationship $(x_1, x_2, x_3, x_4) = (s_1t_1 + s_1u_1 + t_1u_1 - r_1, s_1, t_1, u_1)$ is also a solution to the equation. We notice that $s_1t_1 + s_1u_1 + t_1u_1 - r_1 > u_1 \geq t_1 \geq s_1 \geq r_1$. Therefore, the new solution for x_1 is the new largest value. Repeating this procedure for the variables x_2, x_3, x_4 and so on we can always create a new largest value, hence our largest value tends to infinity and it is larger than N for all real N ■.

That was a mouthful! Go back and read this a few times through, walk away from your computer and walk around with it a bit, it's a confusing method but once you understand it you are golden!

□

Example 5.2.2 (IMO 2007). *Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.*

Solution. Start out with noting that because $\gcd(b, 4ab-1) = 1$, we have:

$$\begin{aligned} 4ab-1 &\mid (4a^2-1)^2 \\ \Longleftrightarrow 4ab-1 &\mid b^2(4a^2-1)^2 \\ \Longrightarrow 4ab-1 &\mid 16a^4b^2 - 8a^2b^2 + b^2 \\ \Longrightarrow 4ab-1 &\mid (16a^2b^2)(a^2) - (4ab)(2ab) + b^2 \\ \Longrightarrow 4ab-1 &\mid (1)(a^2) - (1)(2ab) + b^2 \\ \Longrightarrow 4ab-1 &\mid (a-b)^2 \end{aligned}$$

The last step follows from $16a^2b^2 \equiv (4ab)^2 \equiv 1 \pmod{4ab-1}$ and $4ab \equiv 1 \pmod{4ab-1}$.

Let $(a, b) = (a_1, b_1)$ be a solution to $4ab-1|(a-b)^2$ with $a_1 > b_1$ contradicting $a = b$ where a_1 and b_1 are both positive integers. Assume $a_1 + b_1$ has the smallest sum among all pairs (a, b) with $a > b$, and I will prove this is absurd. To do so, I prove that there exists another solution $(a, b) = (a_2, b_1)$ with a smaller sum. Set $k = \frac{(a-b_1)^2}{4ab_1-1}$ be an equation in a. Expanding this we arrive at

$$\begin{aligned} 4ab_1k - k &= a^2 - 2ab_1 + b_1^2 \\ \Longrightarrow a^2 - a(2b_1 + 4b_1k) + b_1^2 + k &= 0 \end{aligned}$$

This equation has roots $a = a_1, a_2$ so we can now use Vietas on the equation to attempt to prove that $a_1 > a_2$. First, we must prove a_2 is a positive integer. Notice that from $a_1 + a_2 = 2b_1 + 4b_1k$ via Vietas hence a_2 is an integer. Assume that a_2 is negative or zero. If a_2 is zero or negative, then we would have

$$a_1^2 - a_1(2b_1 + 4b_1k) + b_1^2 + k = 0 \geq b^2 + k$$

absurd. Therefore, a_2 is a positive integer and (a_2, b_1) is another pair that contradicts $a = b$. Now, $a_1a_2 = b_1^2 + k$ from Vieta's. Therefore, $a_2 = \frac{b^2+k}{a_1}$. We desire to show that $a_2 < a_1$.

$$\begin{aligned} a_2 &< a_1 \\ \Longleftrightarrow \frac{b_1^2+k}{a_1} &< a_1 \\ \Longleftrightarrow b_1^2 + \frac{(a_1-b_1)^2}{4a_1b_1-1} &< a_1^2 \\ \Longleftrightarrow \frac{(a_1-b_1)^2}{4a_1b_1-1} &< (a_1-b_1)(a_1+b_1) \\ \Longleftrightarrow \frac{(a_1-b_1)}{4a_1b_1-1} &< a_1+b_1 \end{aligned}$$

Notice that we can cancel $a_1 - b_1$ from both sides because we assumed that $a_1 > b_1$. The last inequality is true because $4a_1b_1 - 1 > 1$ henceforth we have arrived at the contradiction that $a_1 + b_1 > a_2 + b_1$. Henceforth, it is impossible to have $a > b$ (our original assumption) and by similar logic it is impossible to have $b > a$ forcing $a = b$ □. □

Example 5.2.3 (Classical)**.** *Let x and y be positive integers such that xy divides $x^2 + y^2 + 1$. Prove that*

$$\frac{x^2 + y^2 + 1}{xy} = 3.$$

Solution. Let $(x, y) = (x_1, y_1)$ be a solution such that $x + y$ is minimal and $\frac{x^2+y^2+1}{xy} = k \neq 3$. WLOG let $x_1 \geq y_1$ (because the equation is symmetric). However, if $x_1 = y_1$, then we must have $\frac{2x_1^2+1}{x_1^2} = 2 + \frac{1}{x_1^2} = k$ and since k is a positive integer, $x_1 = y_1 = 1$ which gives $k = 3$ but we are assuming $k \neq 3$ so hence $x_1 \neq y_1$ and $x_1 \geq y_1 + 1$ (we will use this later). I will prove that we are able to find another solution (x_2, y_1) with $x_2 + y_1 < x_1 + y_1$ forcing $k = 3$ since it contradicts the assumption that $x + y$ is minimal.

$$\begin{aligned} \frac{x^2 + y_1^2 + 1}{xy_1} &= k \\ \implies x^2 - x(ky_1) + y_1^2 + 1 &= 0 \end{aligned}$$

This equation is solved when $x = x_1, x_2$. We will now prove that x_2 is a positive integer. Notice that $x_1 + x_2 = ky_1$ therefore x_2 is an integer. Also from Vietas, $x_1x_2 = y_1^2 + 1 > 0 \implies x_2 > 0$ from $x_1 > 0$. Therefore, x_2 is a positive integer and (x_2, y_1) is another pair that contradicts the $\frac{x^2+y^2+1}{xy} = 3$ statement. Using $x_1x_2 = y_1^2 + 1$, we arrive at $x_2 = \frac{y_1^2+1}{x_1}$. We desire $x_2 < x_1$.

$$\begin{aligned} x_2 &< x_1 \\ \iff \frac{y_1^2 + 1}{x_1} &< x_1 \\ \iff y_1^2 + 1 &< x_1^2 \\ \text{but} x_1^2 \geq (y_1 + 1)^2 = y_1^2 + 2y + 1 &> y_1^2 + 1 \end{aligned}$$

Therefore, $x_2 < x_1$ and we have $y_1 + x_2 < y_1 + x_1$ contradicting our initial assumption, and hence $k = 3$ □. □

Example 5.2.4. *Let a, b be positive integers with $ab \neq 1$. Suppose that $ab-1$ divides a^2+b^2. Show that*

$$\frac{a^2+b^2}{ab-1} = 5$$

.

Solution. Let $(a,b) = (a_1, b_1)$ with WLOG $a_1 \geq b_1$ be the pair of integers such that

$$\frac{a^2+b^2}{ab-1} = k \neq 5$$

and $a+b$ is the smallest. If $a_1 = b_1$, then we would have $\frac{2a_1^2}{a_1^2-1} = k = 2 + \frac{2}{2a_1^2}$ which is only an integer when $a_1 = 1$ however $a_1 = b_1 = 1$ gives $a_1 b_1 = 1$ contradicting $ab \neq 1$ and giving zero in the denominator. Therefore $a_1 \neq b_1$ and $a_1 \geq b_1 + 1$ (we will use this later). I will show that there exist a pair $(a,b) = (a_2, b_1)$ such that $a_1 > a_2$ contradicting $a_1 + b_1$ being minimal.

$$\begin{aligned}
\frac{a^2+b_1^2}{ab_1-1} &= k \\
\Longrightarrow a^2 + b_1^2 &= kab_1 - k \\
\Longrightarrow a^2 - a(kb_1) + b_1^2 + k &= 0
\end{aligned}$$

We now have a quadratic in terms of a with roots $a = a_1, a_2$. I will now prove a_2 is a positive integer. Notice that $a_1 + a_2 = kb_1$ hence a_2 is an integer and $a_1 a_2 = b_1^2 + k$ gives us that a_2 is positive since $b_1^2 + k$ is positive and a_1 is positive.

We desire to prove that $a_2 < a_1$. From Vietas we have $a_1 a_2 = b_1^2 + k$ hence $a_2 = \frac{b_1^2+k}{a_1}$

$$\begin{aligned}
a_2 &< a_1 \\
\Longleftrightarrow \frac{b_1^2+k}{a_1} &< a_1 \\
\Longleftrightarrow b_1^2 + k &< a_1^2 \\
\Longleftrightarrow \frac{a_1^2+b_1^2}{a_1b_1-1} &< a_1^2 - b_1^2 \\
\Longleftrightarrow \frac{a_1^2+b_1^2}{a_1b_1-1} &< a_1 + b_1 \text{from difference of squares and} a_1 - b_1 \geq 1 \\
\Longleftrightarrow a_1^2 + b_1^2 &< a_1^2 b_1 + a_1 b_1^2 - a_1 - b_1 \\
\Longleftrightarrow a_1 + b_1 &< a_1(a_1b_1 - 1) + b_1(a_1b_1 - b_1)
\end{aligned}$$

If $a_1, b_1 \geq 2$ then this inequality is obviously true and $a_2 < a_1$ contradicting the minimal assumption and $k = 5$. If a_1 or b_1 are equal to 1 then we are not done however and we have to prove that $k = \frac{a_1^2+b_1^2}{a_1b_1-1} = 5$ in this situation. Since $a_1 \geq b_1+1$, let $b_1 = 1$. We therefore have $k = \frac{a_1^2+1}{a_1-1} = a_1+1+\frac{2}{a_1-1}$. Therefore, we must have $a_1-1|2$ or $a_1 = 2, 3$. In both of these cases, we get $k = \frac{2^2+1}{2-1} = \frac{3^2+1}{3-1} = 5$. In both cases we have proved $k = 5$ hence we are done □. □

5.2.1 Exercises

5.1. [Crux] If a, b, c are positive integers such that $0 < a^2 + b^2 - abc \leq c$ show that $a^2 + b^2 - abc$ is a perfect square.

5.2. [IMO 1988] Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Prove that $\frac{a^2+b^2}{ab+1}$ is a perfect square.

5.3 Wolstenholme's Theorem

We begin the number theory section with a problem that highlights a key problem solving technique throughout all of mathematics: experimenting.

Theorem 5.3.1 (Wolstenholme's)**.** *For prime p¿3, express* $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$ *in the form* $\frac{m}{n}$ *for* m, n *relatively prime (meaning they share no common divisors). Prove that* p^2 *divides* m*.*

Proof. (Experimenting)

When solving this problem, a mathematician would first off try to simplify the problem. Instead of showing p^2 divides m, we first off try to show that p divides m. To do this, we start out with some small cases, say $p = 5$. Notice that

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} &= 1 + \frac{12}{24} + \frac{8}{24} + \frac{6}{24} \\ &= \frac{50}{24} = \frac{25}{12} \end{aligned}$$

We see quite clearly that $5^2 \mid 25$. We begin writing down some observations that could help us later in the problem.

- The product of the denominators $(2, 3, 4)$ is relatively prime to 5. In fact, the whole set $\{1, 2, 3, \cdots, p-1\}$ is relatively prime to p so it makes sense that there product is relatively prime to p as well.

- In the set of denominators $(1, 2, 3, 4)$ we have $1 + 4 = 5, 2 + 3 = 5$.

We test out the second observation to see if it is of any use:

$$\left(1+\frac{1}{4}\right)+\left(\frac{1}{2}+\frac{1}{3}\right) \;=\; \frac{5}{4}+\frac{5}{6}=5\left(\frac{1}{4}+\frac{1}{6}\right)$$

We now invoke our first observation, and notice that since $\gcd(4\times 6, 5)=1$, the numerator must be divisible by 5 because there are no factors of 5 in the denominator of $\frac{1}{4}+\frac{1}{6}$.

We try extending this grouping idea to the general case:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1} \;=\; \left(1+\frac{1}{p-1}\right)+\left(\frac{1}{2}+\frac{1}{p-2}\right)+\cdots$$

Notice that

$$1+\frac{1}{p-1}=\frac{p}{p-1}, \qquad \frac{1}{2}+\frac{1}{p-2}=\frac{p}{2(p-2)}, \quad \cdots \quad \frac{1}{j}+\frac{1}{p-j}=\frac{p}{j(p-j)}$$

Therefore we can factor out a power of p and the resulting numerator will be divisible by p since there are no powers of p in the denominator.

Now that we've gotten that taken care of, let's move on to the p^2 problem. The first thought that we have at this point is to see what the remaining denominators are after we factor out this first power of p. Let's analyze what we had when $p=5$:

$$5\left(\frac{1}{4}+\frac{1}{6}\right)=5\left(\frac{10}{24}\right)=5\left(\frac{5}{12}\right)$$

Now, obviously $5^2 \mid m$ in this case. The question that puzzles us now is why is the numerator of $\frac{1}{4}+\frac{1}{6}$ likewise divisible by p? Let's try another example when $p=7$:

$$\begin{aligned}\left(1+\frac{1}{6}\right)+\left(\frac{1}{2}+\frac{1}{5}\right)+\left(\frac{1}{3}+\frac{1}{4}\right) &= 7\left(\frac{1}{6}+\frac{1}{10}+\frac{1}{12}\right)\\ &= 7\left(\frac{120+72+60}{6\times 10\times 12}\right)=7\left(\frac{252}{6\times 10\times 12}\right)\\ &= 7\times 7\times\left(\frac{36}{6\times 10\times 12}\right)\end{aligned}$$

Whatever the resulting expression is, it will not contribute any powers of 7 therefore the resulting numerator is divisible by 7^2. But why is $120+72+60$ divisible by 7? Going through two base cases has not yielded anything yet. We could go through and try $p=11$, but I'll spear the reader this. When we hit a dead end in a math problem, we try a different technique. We go back to the general case and remember the equation

$$\frac{1}{j}+\frac{1}{p-j}=\frac{p}{j(p-j)}$$

After factoring out a p, we think "what if we consider the resulting expression mod p". Our study of inverses in the prerequisites tells us that we can indeed consider this expression mod p. Notice that

$$j(p-j)\equiv j(-j)\equiv -j^2 \pmod p$$

Whoah! This expression seems extremely useful. Since $\frac{1}{j}+\frac{1}{p-j}=\frac{p}{j(p-j)}$ uses up two terms, and we want for j to take on all the residues mod p, we must multiply the above expression by 2. For example, $\frac{p}{1(p-1)}\equiv -1^2$, but we want both the -1^2 and the $-(p-1)^2$ terms. Therefore, we now have

$$2\left[\left(\frac{1}{1}+\frac{1}{p-1}\right)+\left(\frac{1}{2}+\frac{1}{p-2}\right)+\cdots\right]\equiv -\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(p-1)^2}\right) \pmod p$$

Since $\gcd(2,p)=1$, it is suficient to show that

$$-\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(p-1)^2}\right)\equiv 0 \pmod p$$

Again, we don't know exactly how to proceed, but we think we have hit the right idea. Let's substitute $p=5$ into the new expression and see what results. What we desire to show is

$$-\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\right)\equiv 0 \pmod 5$$

We think back to what frations mean modulo 5. Fractions are the same thing as inverses, so let's think about the set of inverses modulo 5. The set we are looking at is $\{1^{-1},2^{-1},3^{-1},4^{-1}\} \pmod 5$. Notice that $1^{-1}\equiv 1 \pmod 5, 2^{-1}\equiv 3 \pmod 5, 3^{-1}\equiv 2 \pmod 5, 4^{-1}\equiv 4 \pmod 5$ therefore we have

$$\{1^{-1},2^{-1},3^{-1},4^{-1}\}\equiv\{1,3,2,4\} \pmod 5$$

Notice that this is the same reordered set as $\{1,2,3,4\}$. Therefore, we arrive at

$$-\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\right)\equiv -\left(1^2+2^2+3^2+4^2\right) \pmod 5$$

The resulting expression is divisible by 5 (check it yourself!) Now, we move onto the general case. We desire to show that

$$\{1^{-1}, 2^{-1}, 3^{-1}, \cdots, (p-1)^{-1}\} \equiv \{1, 2, \cdots, (p-1)\} \pmod{p}$$

Now, how would we do this? If we could show that no two numbers in the first set are the same, then the two sets will be congruent. This inspires us to use **proof by contradiction**, where we assume something is true, then show that this is actually a contradiction. Assume that

$$a^{-1} \equiv b^{-1} \pmod{p}, a \not\equiv b \pmod{p}$$

However, multiply both sides of the equation by ab to result in

$$\begin{aligned} aa^{-1}b &\equiv abb^{-1} \pmod{p} \\ a &\equiv b \pmod{p} \end{aligned}$$

Contradiction! Therefore, the two sets must be the same and in general we have

$$-\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(p-1)^2}\right) \equiv -\left(1^2 + 2^2 + 3^2 + \cdots + (p-1)^2\right) \pmod{p}$$

Now, we use the identity

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

To finally arrive at

$$-\left(1^2 + 2^2 + 3^2 + \cdots + (p-1)^2\right) = -\left(\frac{(p-1)(p)(2p-1)}{6}\right)$$

Since $\gcd(p, 6) = 1$ (since $p \geq 5$), the numerator of the expression is divisible by p implying that $p^2 \mid m$. $\square$

That was a mouthful! Throughout this problem, we explored many useful problem solving strategies.

- **Weaken the problem:** When you have no idea how to attack a problem initially, try weakening the condition. In this case, we wanted to show that $p^2 \mid m$, so we tried tos how $p \mid m$, which provided motivation for grouping the sum.

- **Experimenting:** We analyzed simple cases such as $p = 5$ and $p = 7$ to see how would we would go about the weakened version of the problem. We wrote down some observations, and figured out how to solved the weakened problem.

- **Writing out the general form:** When our experimentation failed to solve the general case, we went back to the general form of the equation we found before. Doing so helped us relate the problem to the sum of the squares of inverses.

- **Back to the drawing board:** While the new expression looked more helpful, we had no idea how to deal with inverses. Therefore, we experimented with inverses a bit to make the key observation that the set of inverses mod p and the set of integers mod p (excluding 0) are the same.

- **Prove your lemmas:** While our observation looked extremely helpful, we had to rigorously proof our lemma. In mathematical proof writing, you cannot take statements for granted, you have to prove them. Thankfully the lemma was relatively simple to prove.

Here is a formal writeup of the above proof. The reason that I did not show this formal argument immediately, is many people are left scratching their heads thinking "I understand this, but how did the author come up with this?" Also, to be fully rigorous we must use sigma notation, which is likely confusing to some readers. You can read the following proof lightheartedly, it is included to show the reader how to write up a formal proof of their own once they solve an exciting math problem.

Proof. (Rigorous)

We group the terms as such:

$$\begin{aligned} 2\sum_{i=1}^{p-1}\frac{1}{i} &= \sum_{i=1}^{p-1}\left(\frac{1}{i}+\frac{1}{p-i}\right) \\ &= \sum_{i=1}^{p-1}\left(\frac{p}{i(p-i)}\right) \end{aligned}$$

Since $\gcd(2,p)=1$, we now desire to show that

$$\sum_{i=1}^{p-1}\frac{1}{i(p-i)} \equiv 0 \pmod{p}$$

Notice that $i(p-i) \equiv -i^2$, therefore

$$\sum_{i=1}^{p-1}\frac{1}{i(p-i)} \equiv -\sum_{i=1}^{p-1}\frac{1}{i^2} \pmod{p}.$$

Lemma. All inverses mod a prime are distinct, i.e. $a^{-1} \equiv b^{-1} \pmod{p} \iff a \equiv b \pmod{p}$

Proof.

$$\begin{aligned} a^{-1} &\equiv b^{-1} \pmod{p} \\ (ab)\, a^{-1} &\equiv (ab)\, b^{-1} \pmod{p} \\ a &\equiv b \pmod{p} \end{aligned}$$

□

Therefore

$$\{1^{-1}, 2^{-1}, 3^{-1}, \cdots, (p-1)^{-1}\} \equiv \{1, 2, 3, \cdots, (p-1)\} \pmod{p}$$

Therefore

$$\begin{aligned} -\sum_{i=1}^{p-1} \frac{1}{i^2} \pmod{p} \quad &\equiv \quad -\sum_{i=1}^{p-1} i^2 \\ &= \quad \frac{(p-1)p(2p-1)}{6} \equiv 0 \pmod{p} \end{aligned}$$

Since $\gcd(p, 6) = 1$. We are now done. □

Notice how while the above proof is extremely concise, we have no motivation for why we broke the sum up as such, or how we would have thought of the above solution alltogether!

Here is a similar problem, which is much harder than this above theorem. We again include full motivation and a fully rigorous solution.

Example 5.3.1. *(IberoAmerican Olympiad) Let $p > 3$ be a prime. Prove that if*

$$\sum_{i=1}^{p-1} \frac{1}{i^p} = \frac{n}{m}$$

with $(n, m) = 1$ then p^3 divides n.

Proof. (Experminenting) Incomplete. □

Proof. (Rigorous) This is equivalent to wishing to prove that

$$2\sum_{i=1}^{p-1} \frac{1}{i^p} \equiv 0 \pmod{p^3}$$

treating each of the numbers as inverses and using $\gcd(p, 2) = 1$.

We now notice $\frac{1}{j^p} + \frac{1}{(p-j)^p} = \frac{(p-j)^p + j^p}{j^p(p-j)^p}$. Therefore

$$2\sum_{i=1}^{p-1} \frac{1}{i^p} = \sum_{i=1}^{p-1} \frac{(p-i)^p + i^p}{i^p(p-i)^p}$$

By lifting the exponent $v_p\left((p-i)^p + i^p\right) = v_p\left(p - i + i\right) + v_p(p) = 2$. Therefore we can factor a p^2 out of the numerator to give us

$$\begin{aligned}\sum_{i=1}^{p-1} \frac{(p-i)^p + i^p}{i^p(p-i)^p} = p^2 \sum_{i=1}^{p-1} \frac{\frac{(p-i)^p + i^p}{p^2}}{i^p(p-i)^p} &\equiv 0 \pmod{p^3} \\ \iff \sum_{i=1}^{p-1} \frac{\frac{(p-i)^p + i^p}{p^2}}{i^p(p-i)^p} &\equiv 0 \pmod{p}\end{aligned}$$

Notice that $i \equiv -(p-i) \pmod{p}$ so hence we now need

$$\sum_{i=1}^{p-1} \frac{\frac{(p-i)^p + i^p}{p^2}}{i^{2p}} \equiv 0 \pmod{p}$$

since we can take the negative out of the denominator.

We consider

$$\sum_{i=1}^{p-1} \frac{\frac{(\frac{p+1}{2})^p + (\frac{p-1}{2})^p}{p^2}}{i^{2p}} + \sum_{i=1}^{p-1} \frac{\frac{i^p + (p-i)^p - (\frac{(p+1)}{2})^p - (\frac{(p-1)}{2})^p}{p^2}}{i^{2p}}$$

We desire to show each part is divisible by p. For the first sum notice that we want $\sum_{i=1}^{p-1} \frac{1}{i^{2p}} \equiv \sum_{i=1}^{p-1} i^{2p} \pmod{p}$ since each element from 1 to $p-1$ has a distinct inverse including $1^{-1} \equiv 1 \pmod{p}$ and $(p-1)^{-1} \equiv (p-1) \pmod{p}$.

By Fermat's Little Theorem we have

$$\sum_{i=1}^{p-1} i^{2p} \equiv \sum_{i=1}^{p-1} i^2 \pmod{p} \equiv \frac{(p-1)(p)(2p-1)}{6} \equiv 0 \pmod{p}$$

since $p \neq 2, 3$.

Now we desire to prove that $\frac{i^p + (p-i)^p - \left(\frac{(p+1)}{2}\right)^p - \left(\frac{p-1}{2}\right)^p}{p^2} \equiv 0 \pmod{p}$. Since $\gcd(p, 2) = 1$ we want

$$\begin{aligned}2^p i^p + 2^p(p-i)^p - (p+1)^p - (p-1)^p &\equiv 0 \pmod{p^3} \\ 2^p\left[i^p + \binom{p}{1}p(-i)^{p-1} + (-i)^p\right] - \left[p\binom{p}{1} + 1 + p\binom{p}{1}(-1)^{p-1} + (-1)^p\right] &\equiv 0 \pmod{p^3} \\ 2^p\left[p^2\right] - 2p^2 &\equiv 0 \pmod{p^3}\end{aligned}$$

The second step follows from the fact that $p^2\binom{p}{2} \equiv 0 \pmod{p^3}$ and every $i \geq 3$ we have $p^i\binom{p}{i} \equiv 0 \pmod{p^3}$, and the third step follows from p being odd.

Now, $p^2\left(2^p - 2\right) \equiv 0 \pmod{p^3}$ is true by Fermat's Little Theorem therefore we are done. □

5.4 Bonus Problems

Example 5.4.1 (IMO 1989)**.** *Prove that for all n we can find a set of n consecutive integers such that none of them is a power of a prime number.*

Solution. I claim that the set $\{(2n+2)!+2, (2n+2)!+3, \cdots, (2n+2)!+n+1\}$ satisfies the problem statement. Notice that

$$(2n+2)!+2 = 2\left[1 + \frac{(2n+2)!}{2}\right].$$

Since $\frac{(2n+2)!}{2} \equiv 0 \pmod 2$ it follows that $1 + \frac{(2n+2)!}{2} \equiv 1 \pmod 2$; henceforth it is impossible for $(2n+2)!+2$ to be a power of a prime number because it has an even and an odd factor.

Next, notice that

$$(2n+2)!+k = k\left[1 + \frac{(2n+2)!}{k}\right].$$

Because $2 \leq k \leq n+1$ we must have $(2n+2)! \equiv 0 \pmod{k^2}$ or hence $\frac{(2n+2)!}{k} \equiv 0 \pmod k$. Therefore $1 + \frac{(2n+2)!}{k} \equiv 1 \pmod k$. However, since $(2n+2)!+k$ is divisible by k it must be a perfect power of k, but since $1 + \frac{(2n+2)!}{k}$ is not a perfect power of k it follows that $(2n+2)!+k$ is not a perfect power of a prime. □

Motivation. The way that we arrived at this set may be a bit confusing to the reader so I explain my motivation. When I see a problem like this I instantly think of factorials. The reason behind this is that if I asked to find a set of n consecutive integers none of which are prime, I would use the set

$$\{(n+1)!+2, (n+1)!+3, \cdots, (n+1)!+n+1\}$$

for $n \geq 1$. The reason behind this is we are looking for numbers that have two divisors.

In this case we are looking for numbers that have a divisor divisible by k and another that is not divisible by k. I noticed that looking in the prime factorization of $(2n)!$ that we can find two factors of any number k with $2 \leq k \leq n$. Therefore looking at $(2n!) + k$ we can find that this number is divisible by k but when we factorize it we are left with a term that is $1 \pmod{k}$. $\square$

(Note: The following example includes some intense notation. For this reason we have included a table for $f_2(x)$ and $g_2(x)$ in the "motivation" section in hopes for the reader to better understand the notation.)

Example 5.4.2 (USA TSTST 2013). *Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(n+1) = f(n) + 2^{f(n)}$ for every positive integer n. Prove that $f(1), f(2), \ldots, f(3^{2013})$ leave distinct remainders when divided by 3^{2013}.*

Solution. Define $f(x)$ to be the same as in the problem. Define a function $f_n(x)$ such that $0 \leq f_n(x) \leq 3^n - 1$ and $f_n(x) \equiv f(x) \pmod{3^n}$. Next, define $g_n(x)$ such that $0 \leq g_n(x) \leq \phi(3^n) - 1$ and $g_n(x) \equiv f(x) \pmod{\phi(3^n)}$. We re-write the problem statement as if $x, y \in \{1, 2, \cdots, 3^n\}$ we have $f_n(x) = f_n(y) \iff x = y$ (i.e. $f_n(x)$ is distinct). A few nice properties of these:

1. By Chinese remainder theorem we have $f_{n-1}(x) \equiv f_n(x) \equiv g_n(x) \pmod{3^{n-1}}$.
2. By Euler's totient we have $f_n(x) \equiv 2^{g_n(x-1)} + f_n(x-1) \pmod{3^n}$.
3. From (1) and (2) along with Chinese Remainder Theorem we have $g_n(x) \equiv 2^{g_n(x-1)} + g_n(x-1) \pmod{\phi(3^n)}$

I claim that $f_n(x) = f_n(y) \iff x \equiv y \pmod{3^k}$. This is to say that the values of $f_n(x)$ are distinct when $x \in \{1, 2, \cdots, 3^n\}$ and we have $f_n(x)$ has a period of 3^n (i.e $f_n(x) = f_n(x + 3^n m)$ where m is a positive integer.

Base case: We have $f_1(1) = 1, f_1(2) = 0, f_1(3) = 2$ therefore the problem statement holds for $k = 1$. Notice that $g_1(x) = 1$ therefore $f_1(x) \equiv 2 + f_n(x-1) \pmod 3$ from proposition (2). Therefore it is clear that $f_1(x) = f_1(x + 3m)$.

Inductive hypothesis: $f_n(x) = f_n(y) \iff x \equiv y \pmod{3^n}$ holds for $n = k$. I claim it holds for $n = k + 1$.

From (1) we have $g_{k+1}(x) \equiv f_k(x) \pmod{3^k}$. Therefore we have

$$g_{k+1}(x) \equiv g_{k+1}(y) \pmod{3^k} \iff x \equiv y \pmod{3^k}$$

Since $\phi(3^{k+1}) = 2 \cdot 3^k$ and $g_{k+1}(x)$ is odd we have $g_{k+1}(x) = g_{k+1}(x + 3^k m)$ for positive integers m. This means that we can separate $g_{k+1}(x)$ into three "groups" that have length 3^k and are ordered $g_{k+1}(1), g_{k+1}(2), \cdots, g_{k+1}(3^k)$.

Because $f_k(x) \equiv g_{k+1}(x) \equiv f_{k+1}(x) \pmod{3^k}$ we can separate $f_{k+1}(x)$ into three groups all of whose elements correspond to $f_k(x)$ mod 3^k. We now have

$$f_{k+1}(x) \equiv f_{k+1}(y) \pmod{3^k} \iff x \equiv y \pmod{3^k}$$

We must now prove that

- $f_{k+1}(x) = f_{k+1}(x + 3^{k+1}m)$ where m is a positive integer.
- $f_{k+1}(x) \not\equiv f_{k+1}(x + 3^k) \pmod{3^{k+1}} \not\equiv f_{k+1}(x + 2 \cdot 3^k) \pmod{3^{k+1}}$.

Both of these two statements boil down to (2). We notice that we have

$$\begin{aligned} f_{k+1}(x + 3^k) &\equiv 2^{g_{k+1}\left(x+3^k-1\right)} + f_{k+1}\left(x + 3^k - 1\right) \pmod{3^{k+1}} \\ \implies f_{k+1}(x + 3^k) &\equiv 2^{g_{k+1}\left(x+3^k-1\right)} + 2^{g_{k+1}\left(x+3^k-2\right)} + \cdots + 2^{g_{k+1}(x)} + f_{k+1}(x) \pmod{3^{k+1}} \end{aligned}$$

We now notice that $g_{k+1}(y)$ takes on all of the odd integers from 1 to $\phi\left(3^{k+1}\right) - 1$. This is exactly the number of elements we have above henceforth it happens that

$$\begin{aligned} f_{k+1}(x + 3^k) &\equiv 2^1 + 2^3 + \cdots + 2^{\phi\left(3^{k+1}\right)-1} + f_{k+1}(x) \pmod{3^{k+1}} \\ f_{k+1}(x + 3^k) &\equiv 2\left(\frac{2^{\phi(3^{k+1})} - 1}{3}\right) + f_{k+1}(x) \pmod{3^{k+1}} \end{aligned}$$

.

Via lifting the exponent we have

$$v_3\left(2^{\phi(3^{k+1})} - 1\right) = k + 1$$

Therefore $v_3\left(\dfrac{2^{\phi(3^{k+1})} - 1}{3}\right) = k$ henceforth $f_{k+1}(x + 3^k) \equiv f_{k+1}(x) \pmod{3^k}$ but $f_{k+1}(x + 3^k) \not\equiv f_{k+1}(x) \pmod{3^{k+1}}$. We write $f_{k+1}(x + 3^k) - f_{k+1}(x) = z3^k$ where $\gcd(z, 3) = 1$. Notice that substituting $x = n3^k + x_1$ we arrive at

$$f_{k+1}(x_1 + (n + 1) \cdot 3^k) - f_{k+1}(x_1 + n3^k) \equiv z3^k \pmod{3^{k+1}}$$

from which we can derive

$$f_{k+1}\left(x + a \cdot 3^k\right) - f_{k+1}\left(x + b \cdot 3^k\right) \equiv (a - b)3^k \pmod{3^{k+1}}$$

Now, notice that $f_{k+1}(x+3^k)-f_{k+1}(x) = z3^k$, $f_{k+1}(x+2\cdot3^k)-f_{k+1}(x+3^k) = z3^k$ and $f_{k+1}(x+2\cdot3^k)-f_{k+1}(x) = 2z3^k$ therefore the first part of our list of necessary conditions is taken care of. Now, notice that substituting $a = 3m, b = 0$ into the second equation we arrive at the second condition.

Our induction is henceforth complete. Because the problem statement holds for all n it also holds for 2013 and we are done. $\square$

Motivation. The main motivation I had when solving this problem was to look at the table of $f_2(x)$ and $g_2(x)$. This table was derived using properties (2) and (3). Notice how the other properties hold for this table.

x	$f_2(x)$	$g_2(x)$
1	1	1
2	3	3
3	5	2
4	1	7
5	3	9
6	5	8
7	1	4
8	3	6
9	5	5

□

Example 5.4.3 (IMO 2005)**.** *Determine all positive integers relatively prime to all the terms of the infinite sequence* $2^n + 3^n + 6^n - 1, n \geq 1$

Solution. [[2]]

I prove that for all primes p there exists a term in the sequence that is divisible by p. I claim that when $n = p - 2$ we have $2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \equiv 0 \pmod{p}$.

$$\begin{aligned} & 6\left(2^{p-2} + 3^{p-2} + 6^{p-2} - 1\right) \\ \equiv\ & 3(2^{p-1}) + 2(3^{p-1}) + 6^{p-1} - 6 \\ \equiv\ & 3(1) + 2(1) + 6(1) - 6 \equiv 0 \pmod{p} \end{aligned}$$

Therefore when $p \neq 2, 3$ it follows that

$$2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \equiv 0 \pmod{p}$$

When $p = 2$ we notice that $n = 1$ gives $2^1 + 3^1 + 6^1 - 1 = 10 \equiv 0 \pmod{2}$ and when $p = 3$ that $n = 2$ gives $2^2 + 3^2 + 6^2 - 1 = 48 \equiv 0 \pmod{3}$.

In conclusion notice that all primes divide a term in the sequence hence the only positive integer relatively prime to every term in the sequence is 1.

□

Motivation. The motivation behind this problem is noticing that most likely every prime divides into at least one term and then trying to find which value of n generates this. The motivation fo rhte choice of n stems from noticing that $2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 \equiv 0 \pmod{p}$. This is unfortunately not a fully rigorous proof as most olympiads do not accept working in fractions when we think about modulos (even though it is a fully legitimate method and is in fact necessary to prove some theorems). □

Example 5.4.4 (IMO 1971)**.** *Prove that we can find an infinite set of positive integers of the form $2^n - 3$ (such that n is a positive integer) every pair of which are relatively prime.*

Solution. [[15],[3]]

We use induction. Our base case is when a set has 2 elements and that is done by $\{5, 13\}$. Let a set have N elements and I prove it is always possible to construct a $N + 1$ element set with the new element larger than all the previous elements. Let all the distinct primes that divide the least common multiple of the N elements be $p_1, p_2, \cdots, p_k$. Denote the new element that we add to the set to be $2^x - 3$. We desire to have $2^x - 3 \not\equiv 0 \pmod{p_j}$ for $1 \le j \le k$. Since $\gcd(p_j, 2) = 1$ we have $2^{p_j - 1} \equiv 1 \pmod{p_j}$. Therefore letting

$$x = \prod_{i=1}^{k} (p_i - 1)$$

to give us $2^x - 3 \equiv -2 \pmod{p_j}$ and since $\gcd(p_j, 2) = 1$ we have $2^x - 3 \not\equiv 0 \pmod{p_j}$ as desired.

Notice that this process is always increasing the newest member of the set therefore we may do this process an infinite amount of times to give us an infinite set.

□

Motivation. The motivation behind using induction is to think of how you can construct an infinite set. It would be quite tough to build an infinite set without finding a way to construct a new term in the sequence which is essentially what we do here. □

Example 5.4.5. *(IMO Shortlist 1988) A positive integer is called a double number if its decimal representation consists of a block of digits, not commencing with 0, followed immediately by an identical block. So, for instance, 360360 is a double number, but 36036 is not. Show that there are infinitely many double numbers which are perfect squares.*

Solution. Let $C(n)$ be this function and notice that when $n = \sum_{i=0}^{k-1}(10^i a_i)$ where $0 \le a_m \le 9$ when $m \ne k-1$ and $1 \le a_{k-1} \le 9$, we have

$$C(n) = (10^k + 1)\sum_{i=0}^{k-1}(10^i a_i)$$

We set

- $10^k + 1 = 49 p_1^{e_1} \cdots p_m^{e_m}$
- $\displaystyle\sum_{i=0}^{k-1}(10^i a_i) = 36 p_1^{e_1} \cdots p_m^{e_m}$

Obviously $C(n)$ is a perfect square in this case. Also since

$$10^{k-1} < \sum_{i=0}^{k-1}(10^i a_i) = \frac{36}{49}(10^k + 1) < 10^k$$

it follows that $1 \le a_{k-1} \le 9$. It is left to prove that there are infinite k such that $49|(10^k + 1)$. Noticing that $\mathrm{ord}_{49}(10) = 42$, we see that $k = 42x + 21$ satisfies the condition $10^k + 1 \equiv 0 \pmod{49}$ hence we have constructed infinitely many double numbers which are perfect squares. $\square$

A

Notation

This text was written primarily for any audience who enjoys number theory and wants to learn new problem solving skills. In this text, we will attack many hard problems using many different methods and tips in number theory. In this text, we attack many hard math problems using simple methods and formulae. Each section begins with a theorem or general idea, along with a fully rigorous proof. By the end of this text, I hope the reader has mastered the method of induction. Each section is then filled with problems off of the main idea of the section. Instead of including many computational problems, we begin with a few "easier" problems and then dig right into olympiad problems. While this may be hard or challenging to those just getting acquainted with mathematics, through personal experience, this is the best way to learn number theory. I highly recommend the reader spends time on each and every problem before reading the given solution. If you do not solve the problem immediately, do not fret, it took me a very long time to solve most of the problem myself.

Spend your time and struggle through the problems, and enjoy this text!

A.0.1 Sets

- The real numbers $\mathbb{R}$ are any positive or negative number including 0, such as $1, 1+\sqrt{2}, -\pi, e$, etc
- The integers $\mathbb{Z}$ are defined as the integers:

$$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

  $\mathbb{Z}^+$ denotes the positive integers $\{1, 2, 3, \cdots\}$ while $\mathbb{Z}^-$ denotes the negative integers $\{\cdots, -3, -2, -1\}$.
- The natural numbers $\mathbb{N}$ are defined as the positive integers or $\mathbb{Z}^+$. The natural numbers including 0 are defined as $\mathbb{N}^0$.

- The rational numbers $\mathbb{Q}$ are defined as the ratio of two integers, such as $\frac{2}{3}$ or $\frac{17}{29}$.

- The complex numbers $\mathbb{C}$ are defined as $a + bi$ where $a, b \in \mathbb{R}$.

- The set of polynomials with integer coefficients is defined as $\mathbb{Z}[x]$. For example, $x^3 - 19x^{18} + 1 \in \mathbb{Z}[x]$, however, $x^2 - \pi x \notin \mathbb{Z}[x]$.

- The set of polynomials with rational coefficients is defined as $\mathbb{Q}[x]$. For example, $x^2 - \frac{1}{2}x \in \mathbb{Q}[x]$.

We say that a divides b if $\frac{b}{a}$ is an integer. For example, 4 divides 12 since $\frac{12}{4} = 3$, however, 4 does not divide 13 since $\frac{13}{4} = 3.25$. We write a divides b as $a \mid b$. In this, b is also a multiple of a. **In this text, when we say "divisors" we assume positive divisors.** When considering divisors of natural n, we only have to work up to $\sqrt{n}$. The reason for this is if $n = ab$ then we obviously cannot have $a, b > \sqrt{n}$.

Induction is a proof technique used often in math. As it can be tricky to those who are understanding it for the first time, we begin with an example problem and explain the method of induction as we solve this problem.

Example A.0.1. *Show that for all natural n, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.*

Solution. In induction, we first off have to show a statement holds for a base case, typically $n = 1$. In this case,

$$1 = \frac{1 \times 2}{2}$$

so the base case holds. We now show that if the problem statement holds for $n = k$, then it holds for $n = k+1$. This essentially sets off a chain, where we have

$$n = 1 \implies n = 2 \implies n = 3 \implies \cdots$$

The reason we have to show the base case is because it is the offseter of the chain. Because of this reason, we can think of induction as a chain of dominoes. Once we knock down the first domino, and show that hitting a domino will knock down the proceeding domino, we know all the dominoes will be knocked down. Our inductive hypothesis is that the problem statement holds for $n = k$, or henceforth

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

We now need to show that it holds for $n = k+1$ or we need to show that $1+2+3+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}$. Now, notice that

$$\begin{aligned} 1+2+3+\cdots+(k+1) &= (1+2+3+\cdots+k)+k+1 \\ &= \frac{k(k+1)}{2}+k+1 = \frac{(k+1)(k+2)}{2} \end{aligned}$$

As desired. Therefore, we have completed our induction. □

Theorem A.0.1 (Induction)**.** *Let's say we have a statement $P(n)$ that we wish to show holds for all natural n. It is sufficient to show the statement holds for $n = 1$ and that $P(k) \implies P(k+1)$ for natural k, then the statement is true for all natural n.*

NOTE: *The statement $P(k) \implies P(k+1)$ means that if $P(k)$ is true (meaning the statement holds for $n = k$), then $P(k+1)$ is true. This is used for ease of communication.*

Proof. We use the **well ordering principle**. The well ordering principle states that every set has a smallest element. In this case, assume that for sake of contradiction, $P(n)$ is not true for some $n = x \in S$. Let y be the smallest element of S and since $y > 1$ (from us showing the base case), we have $y - 1 \geq 1$. Therefore $P(y-1)$ is true. We also know that $P(k) \implies P(k+1)$. Therefore, $P(y-1) \implies P(y)$, contradiction. □

Theorem A.0.2 (Strong induction)**.** *For a statement $P(n)$ that we wish to show holds for all natural n, it is sufficient to show a base case ($n = 1$) and that if $P(n)$ is true for $n \in \{1, 2, 3, \cdots, k\}$ it implies $P(k+1)$ is true.*

Proof. The proof is identical to the above proof verbatum. □

It is assumed the reader has prior knowledge of induction, so this should be review. If induction is still confusing at this point, we recommend the reader reads up on induction as it is vital for this text.

This section includes formulas it is assumed the reader knows.

Theorem A.0.3 (Binomial Theorem)**.** *For n natural,*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Definition A.0.1. The greatest common divisor of two integers a, b is denoted $\gcd(a, b)$. For example, $\gcd(4, 12) = 4$.

Definition A.0.2. The least common multiple of two integers a, b is denoted $\text{lcm}[a, b]$. For example $\text{lcm}[4, 15] = 60$

Definition A.0.3. We define

$$a \equiv b \pmod{c} \iff c \mid a - b$$

For example $13 \equiv 1 \pmod{4}$ since $4 \mid 12$.

Definition A.0.4. A number is said to be prime if the only divisors of the number are 1 and itself. For example, 5 is prime since $1 \mid 5, 2 \nmid 5, 3 \nmid 5, 4 \nmid 5, 5 \mid 5$. On the other hand, 6 is not prime as $1, 2, 3, 6 \mid 6$. A number is said to be composite if n can be expressed in the form ab for a, b being positive integers greater than 1. 1 is said to be neither prime nor composite.

Definition A.0.5. $\longrightarrow\!\longleftarrow$ means "contradiction"

Definition A.0.6. The degree of a polynomial is defined as the highest exponent in its expansion. For example, $\deg(x^3 - 2x^2 + 1) = 3$ and $\deg(-x^2 + x^4 - 1) = 4$.

B

Solutions

Solutions of Chapter 1

1.1 Note that

$$603 = 301 \times 2 + 1$$

Therefore, by the Euclidean Algorithm, we have $\gcd(603, 301) = \gcd(1, 301) = \boxed{1}$.

1.2

$$\begin{aligned} 289 &= 153 \times 1 + 136 \\ 153 &= 136 \times 1 + 17 \\ 136 &= 17 \times 8 + 0. \end{aligned}$$

Therefore $\gcd(153, 289) = \boxed{17}$.

1.3

$$\begin{aligned} 189 &= 133 \times 1 + 56 \\ 133 &= 56 \times 2 + 19 \\ 56 &= 19 \times 2 + 18 \\ 19 &= 18 \times 1 + 1. \end{aligned}$$

Therefore $\gcd(189, 133) = \boxed{1}$.

1.4 Let the two numbers be a, b with $a > b$. Recall that $\gcd(a, b)\text{lcm}[a, b] = ab$, therefore $ab = 384$. Furthermore, $a = b + 8$, therefore we have

$$b(b + 8) = 384 \implies b^2 + 8b - 384 = 0 \implies b = 16.$$

This gives $a = b + 8 = 24$, therefore $a + b = 24 + 16 = \boxed{40}$.

1.5 The first step is to divide the leading term of $a(x)$, x^5, by the leading term of $b(x)$, x^3: $\frac{x^5}{x^3} = x^2$. Therefore

$$x^5 + 4x^2 + 2x = \left(x^3 + 2x^2\right) x^2 + \left(-2x^4 + 4x^2 + 2x\right).$$

The next step is to divide the leading term of $r_1(x)$, $-2x^4$, by the leading term of $b(x)$, x^3: $\frac{-2x^4}{x^3} = -2x$. We add this to the quotient and subtract from the remainder:

$$x^5 + 4x^2 + 2x = \left(x^3 + 2x^2\right)\left(x^2 - 2x\right) + \left(4x^3 + 4x^2 + 2x\right).$$

Finally, we once again divide the leading of $r_2(x)$, $4x^3$ by the leading term of x^3: $\frac{4x^3}{x^3} = 4$. We add this to quotient and subtract from the remainder:

$$x^5 + 4x^2 + 2x = \left(x^3 + 2x^2\right)\left(x^2 - 2x + 4\right) + \left(-4x^2 + 2x\right).$$

Therefore, $q(x) = x^2 - 2x + 4$ and $r(x) = -4x^2 + 2x$. Note that $\deg(r(x)) = 2 < \deg(b(x)) = 3$.

1.6 In order for the improper fraction to not be in lowest terms, we must have $\gcd(N^2 + 7, N + 4) \neq 1$. We find that

$$N^2 + 7 = (N + 4)(N - 4) + 23.$$

Therefore $\gcd(N^2 + 7) = \gcd(N + 4, 23)$ Since 23 is prime, we must have $23 \mid N + 4$. The smallest value of N which works in the specified range is $23 \times 0 + 19 = 19$, and the largest is $23 \times 85 + 19 = 1974$. Therefore, there is a total of $85 + 1 = \boxed{86}$ values of N which work.

1.7

$$\begin{aligned}
\gcd(21n + 4, 14n + 3) &= \gcd(7n + 1, 14n + 3) \\
&= \gcd(7n + 1, 14n + 3 - 2(7n + 1)) \\
&= \gcd(7n + 1, 1) \\
&= 1.
\end{aligned}$$

1.8

$$\begin{aligned}
x^4 - x^3 &= \left(x^3 - x\right)(x - 1) + \left(x^2 - x\right) \\
x^3 - x &= \left(x^2 - x\right)(x + 1).
\end{aligned}$$

Therefore, $\gcd(x^4 - x^3, x^3 - x) = \boxed{x^2 - x}$.

1.9 We use the division algorithm to divide ax^3+bx^2+1 by x^2-x-1. First off, divide the leading terms to get $\frac{ax^3}{x^2}=ax$:

$$ax^3+bx^2+1=\left(x^2-x-1\right)(ax)+\left[(a+b)x^2+ax+1\right].$$

Next, divide the leading term of the remainder by the leading term of the dividend to get $\frac{(a+b)x^2}{x^2}=a+b$. Add this to the quotient:

$$ax^3+bx^2+1=\left(x^2-x-1\right)(ax+a+b)+\left[(2a+b)x+a+b+1\right].$$

In order for x^2-x-1 to divide ax^3+bx^2+1, the remainder must be 0. Therefore we must have

$$\begin{cases}2a+b=0\\ a+b+1=0\end{cases}.$$

From the first equation, we hae $b=-2a$. Substituting this into the second equation gives $a+(-2a)+1=0 \implies a=\boxed{1}$ and $b=\boxed{-2}$.

1.10 We use induction. For a base case, note that $\gcd(F_1,F_2)=\gcd(1,1)=1$ and $\gcd(F_2,F_3)=\gcd(1,2)=1$.

For the inductive hypothesis part, assume that $\gcd(F_m,F_{m+1})=1$. We now show that this implies that $\gcd(F_{m+1},F_{m+2})=1$, completing the induction. By the Euclidean Algorithm, note that

$$\gcd(F_{m+1},F_{m+2})=\gcd(F_{m+1},F_{m+2}-F_{m+1}).$$

By definition of Fibonacci numbers, we have $F_{m+2}=F_{m+1}+F_m \implies F_{m+2}-F_{m+1}=F_m$. Therefore,

$$\gcd(F_{m+1},F_{m+2})=\gcd(F_{m+1},F_m)=1$$

by the inductive hypothesis. Therefore, we have shown that $m \implies m+1$ and by induction, we are done.

1.11 Let the two relatively prime numbers be a and b. Assume for the sake of contradiction that $\gcd(ab,a+b)\neq 1$, meaning that there exists a prime p such that $p \mid ab$ and $p \mid a+b$. Recall that by Euclid's Lemma,

$$p \mid ab \implies p \mid a \text{ or } p \mid b.$$

However, in the case that $p \mid a$, since we also know that $p \mid a+b$, we must have $p \mid b$, contradicting the fact that a and b are relatively prime. The same argument holds in the case that $p \mid b$. Therefore, we have arrived at a contradiction, and we must have $\gcd(ab,a+b)=1$.

1.12

$$\begin{aligned} 97 &= 5 \times 19 + 2 \\ 5 &= 2 \times 2 + 1. \end{aligned}$$

Running these steps in reverse, we find that

$$1 = 5 - 2 \times 2 = 5 - 2 \times (97 - 5 \times 19) = 5 \times 39 + 97 \times (-2).$$

Therefore $x = 39$ and $y = -2$.

1.13

$$\begin{aligned} 1110 &= 1011 \times 1 + 99 \\ 1011 &= 99 \times 10 + 21 \\ 99 &= 21 \times 4 + 15 \\ 21 &= 15 \times 1 + 6 \\ 15 &= 6 \times 2 + 3. \end{aligned}$$

Running these steps in reverse gives

$$\begin{aligned} 3 &= 15 - 2 \times 6 \\ &= 15 - 2(21 - 15 \times 1) = 3 \times 15 - 2 \times 21 \\ &= 3 \times (99 - 21 \times 4) - 2 \times 21 = 3 \times 99 - 14 \times 21 \\ &= 3 \times 99 - 14 \times (1011 - 99 \times 10) = 143 \times 99 - 14 \times 1011 \\ &= 143 \times (1110 - 1011) - 14 \times 1011 = 143 \times 1110 - 157 \times 1011. \end{aligned}$$

Therefore $x = 143$ and $y = -157$.

1.14 By the Euclidean Algorithm, note that

$$\begin{aligned} 1691 &= 1349 \times 1 + 342 \\ 1349 &= 342 \times 3 + 323 \\ 342 &= 323 \times 1 + 19 \\ 323 &= 19 \times 17 + 0. \end{aligned}$$

Therefore, $\gcd(1691, 1349) = 19$, and it is impossible for a linear combination of 1691 and 1349 to be equal to 1.

1.15 Note that the pair $(x, y) = (-5, 2)$ satisfies the equation. The problem asks to find all integers x, y, therefore, we need to parameterize it. If (x_1, y_1) is a solution, then so is $(x_1 + 13, y_1 - 5)$ since

$$5(x_1 + 13) + 13(y_1 - 5) = 5x_1 + 13y_1 + (65 - 65) = 1.$$

Therefore, for integer t, all solutions (x, y) are characterized by $\boxed{(-5 + 13t, 2 - 5t)}$.

1.16 Note that

$$n^k - 1 = (n-1)\left(n^{k-1} + n^{k-2} + \cdots + n + 1\right).$$

Therefore,

$$(n-1)^2 \mid n^k - 1 \iff (n-1) \mid \left(n^{k-1} + n^{k-2} + \cdots + n + 1\right).$$

Furthermore, since $n \equiv 1 \pmod{n-1}$, we have

$$n^{k-1} + n^{k-2} + \cdots + n + 1 \equiv 1^{k-1} + 1^{k-2} + \cdots + 1 + 1 \equiv k \pmod{n-1}.$$

Therefore, $(n-1) \mid k$.

1.17 Assume for the sake of contradiction that $\sqrt{2}$ is rational. Therefore, for relatively prime integers m, n, we have $\sqrt{2} = \frac{m}{n}$. Multiply by n on both sides and square in order to get

$$m^2 = 2n^2.$$

Since the right hand side is a multiple of 2, the left hand side must be also, therefore $2 \mid m$. For some integer m_1, set $m = 2m_1$:

$$(2m_1)^2 = 2n^2 \implies 2m_1^2 = n^2.$$

By similar logic, the left hand side is a multiple of 2, therefore the right hand side must be, therefore $2 \mid n$. However, this contradicts the initial assumption that $\frac{m}{n}$ was in lowest terms, since 2 divides both the numerator and denominator. Therefore, we have arrived at a contradiction and are done.

1.18 Assume for the sake of contradiction that $\log_{10}(2)$ is rational. Therefore, for relatively prime integers m, n, we must have $\log_{10}(2) = \frac{m}{n}$. Rewriting this equation, we see that it is equivalent to

$$2^{\frac{m}{n}} = 10.$$

Taking everything to the power of n gives $2^m = 10^n$. Note that $10^n = 2^n 5^n$, therefore, the equation becomes $2^m = 2^n 5^n$. However, this is a contradiction of the Fundamental Theorem of Arithmetic, because the right side has a prime factor of 5 and the left side does not.

1.19 We begin by prime factorizing 2004: $2004 = 2^2 \cdot 3 \cdot 167$. Therefore, $2004^{2004} = 2^{4008} \cdot 3^{2004} \cdot 167^{2004}$. Hence, any divisor of 2004^{2004} will take on the form $m = 2^a 3^b 167^c$. If the divisor m has exactly 2004 positive integers, then this is equivalent to

$$\tau(m) = (a+1)(b+1)(c+1) = 2004 = 2^2 \cdot 3 \cdot 167.$$

We consider the set $\{a+1, b+1, c+1\}$. We begin by figuring out how to place the factors of 167 into the set. We can either give the factor of 167 to $a+1$, $b+1$, or $c+1$, for a total of **3** choices.

Similarly, for the factor of 3, we have a total of **3** choices. On the other hand, for the factor of 2, this is equivalent to placing 2 indistinguishable objects (factors of 2) into 3 bins. Using the method of stars of bars, this can be done in $\binom{2+3-1}{2-1} = \mathbf{6}$ ways. Another way to verify this would be simple casework on the powers of 2 in $\{a+1, b+1, c+1\}$: $(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)$.

In conclusion, there are $3 \cdot 3 \cdot 6 = \boxed{54}$ positive integer divisors of 2004^{2004} that are divisible by 2004 positive integers.

1.20 We proceed with the Euclidean Algorithm over the rationals:

$$\begin{aligned}
x^3 - 1 &= (5x^2 - 1)\left(\frac{1}{5}x\right) + \frac{x}{5} - 1 \\
(5x^2 - 1) &= \left(\frac{x}{5} - 1\right)(25x + 125) + 124 \\
1 &= \frac{5x^2 - 1}{124} - \left(\frac{x}{5} - 1\right)\left(\frac{25}{124}x + \frac{125}{124}\right) \\
1 &= \frac{5x^2 - 1}{124} - \left[x^3 - 1 - \frac{1}{5}x\left(5x^2 - 1\right)\right]\left(\frac{25}{124}x + \frac{125}{124}\right) \\
1 &= \left(5x^2 - 1\right)\left[\frac{1}{124} + \frac{1}{5}x\left(\frac{25}{124}x + \frac{125}{124}\right)\right] - \left(x^3 - 1\right)\left(\frac{25}{124}x + \frac{125}{124}\right) \\
1 &= \left(5x^2 - 1\right)\left(\frac{5}{124}x^2 + \frac{25}{124}x + \frac{1}{124}\right) - \left(x^3 - 1\right)\left(\frac{25}{124}x + \frac{125}{124}\right).
\end{aligned}$$

Therefore,

$$u(x) = \frac{5}{124}x^2 + \frac{25}{124}x + \frac{1}{124} \quad \text{and} \quad v(x) = \frac{-25x}{124} - \frac{125}{124}.$$

1.21 We begin by applying the division algorithm repeatedly.

$$\begin{aligned}
x^8 - 1 &= (x^5 - 1)x^3 + x^3 - 1 \\
x^5 - 1 &= (x^3 - 1)x^2 + x^2 - 1 \\
x^3 - 1 &= (x^2 - 1)x + x - 1 \\
x^2 - 1 &= (x - 1)x + 1.
\end{aligned}$$

We then reverse the order of the division algorithm as follows:

$$\begin{aligned} x-1 &= x^3-1-(x^2-1)x \\ &= x^3-1-\left(x^5-1-(x^3-1)x^2\right)x \\ &= \left(x^3-1\right)\left(1+x^3\right)-\left(x^5-1\right)x \\ &= \left(x^8-1-(x^5-1)x^3\right)\left(1+x^3\right)-\left(x^5-1\right)x \\ &= \left(x^8-1\right)\left(1+x^3\right)+\left(x^5-1\right)\left(-x^6-x^3-x\right). \end{aligned}$$

Hence, polynomials that satisfy the above condition are $u(x) = \boxed{1+x^3}$ and $v(x) = \boxed{-x^6-x^3-x}$.

1.22 I claim the answer to be yes. Note that by the 1.1.5, we have

$$\gcd(x^m-1, x^n-1) = x^{\gcd(m,n)}-1 = x-1,$$

since it is given in the problem statement that m and n are relatively prime. Therefore, using Bezout's Theorem for polynomials, there does indeed exist $u, v \in \mathbb{Q}[x]$ such that $(x^m-1)u(x)+(x^n-1)v(x) = \gcd(x^m-1, x^n-1) = x-1$. In order to find such polynomials, one should follow the method used in the above problem.

1.23 Assume for the sake of contradiction that there exists positive integers a, b for which the above equation was true. Let $\gcd(a, b) = d$. Now, set $a = da_1$ and $b = db_1$ and substitute these into the equation above to get

$$d^n\left(a_1^n-b_1^n\right) \mid d^n\left(a_1^n+b^n\right) \implies a_1^n-b_1^n \mid a_1^n+b_1^n.$$

Now, we know that $\gcd(a_1, b_1) = 1$. However, if the above equation is true, then we must also have

$$\begin{aligned} a_1^n-b_1^n \mid (a_1^n+b_1^n)+(a_1^n-b_1^n) &= 2a_1^n \\ a_1^n-b_1^n \mid (a_1^n+b_1^n)-(a_1^n-b_1^n) &= 2b_1^n. \end{aligned}$$

If $a_1^n-b_1^n$ divides both $2a_1^n$ and $2b_1^n$, it must then divide their greatest common divisor:

$$a_1^n-b_1^n \mid \gcd(2a_1^n, 2b_1^n) = 2.$$

Since $n>1$, we know that $a_1^n-b_1^n \neq 1, 2$, leading to a contradiction.

1.24 Note that the greatest common divisor of the first two terms must divide the entire result. Therefore, we calculate it as follows:

$$\begin{aligned} \gcd(2002+2, 2002^2+2) &= \gcd\left(2002+2, 2002^2+2-(2002+2)(2002-2)\right) \\ &= \gcd\left(2002+2, 2002^2+2-\left(2002^2-4\right)\right) \\ &= \gcd\left(2002+2, 6\right) \\ &= 6. \end{aligned}$$

Therefore, the greatest common divisor of the sequence must divide 6. We now prove that 6 divides every term in the sequence. Clearly, every term in the sequence is even, and furthermore, since $2002 \equiv 1 \pmod{3}$, we have

$$2002^k + 2 \equiv 1^k + 2 \equiv 0 \pmod{3}.$$

Therefore, the answer is $\boxed{6}$.

1.25

$$\begin{aligned} \gcd(n!+1,(n+1)!) &= \gcd(n!+1,(n+1)!-(n+1)(n!+1)) \\ &= \gcd(n!+1,-(n+1)) \\ &= \gcd(n!+1,n+1). \end{aligned}$$

Let p be a prime divisor of $n+1$. Unless $n+1$ is prime, we have

$$p \le n \implies n!+1 \equiv 1 \pmod{p}.$$

When $n+1$ is prime, we have $n!+1 \equiv 0 \pmod{n+1}$ (this fact will later be proved in the Wilson's theorem section). Therefore, the answer is

$$\gcd(n!+1,(n+1)!) = \begin{cases} 1 & \text{if } n+1 \text{ is composite} \\ n+1 & \text{if } n+1 \text{ is prime} \end{cases}.$$

1.26

$$\begin{aligned} \gcd(2^{30^{10}}-2, 2^{30^{45}}-2) &= 2\left[2^{\gcd(30^{10}-1,30^{45}-1)}-1\right] \\ &= 2\left[2^{30^{\gcd(10,45)}-1}-1\right] \\ &= 2\left(2^{30^5-1}-1\right) = 2^{30^5}-2. \end{aligned}$$

Therefore, $x = 30^5$.

1.27 We begin by using the formula for the sum of squares to rewrite the sum:

$$\begin{aligned} 2^2+3^2+\cdots+n^2 &= \left(1^2+2^2+3^2+\cdots+n^2\right)-1^2 \\ &= \frac{n(n+1)(2n+1)}{6}-1 \\ &= \frac{2n^3+3n^2+n-6}{6} \\ &= \frac{(n-1)(2n^2+5n+6)}{6}. \end{aligned}$$

We want to determine when this expression equals p^k for some prime p. We begin by computing the greatest common divisor of $n-1$ and $2n^2+5n+6$. Note that

$$2n^2+5n+6=(n-1)(2n+7)+13.$$

Therefore, $\gcd(2n^2+5n+6, n-1)=\gcd(13, n-1)$. Hence, $n-1$ and $2n^2+5n+6$ can share no common divisors other than 13.

Hence, besides for a few special cases, there will be more than one prime that divides into the expression $\dfrac{(n-1)(2n^2+5n+6)}{6}$. The exception cases are when $n-1$ divides into the denominator, 6. We therefore check $n=2,3,4,7$.

- When $n=2$, then $\frac{(n-1)(2n^2+5n+6)}{6}=4=2^2$.
- When $n=3$, then $\frac{(n-1)(2n^2+5n+6)}{6}=13$.
- When $n=4$, then $\frac{(n-1)(2n^2+5n+6)}{6}=29$.
- When $n=7$, then $\frac{(n-1)(2n^2+5n+6)}{6}=139$.

All of the numbers above are of the form p^k, hence, the answer is $n=\boxed{2,3,4,7}$.

1.28 First off, WLOG let $m>n$. Then we have

$$a^{2^n}+1 \mid a^{2^{n+1}}-1 \mid a^{2^m}-1.$$

The last step follows from the fact that $2^{n+1} \mid 2^m$.

Let $a^{2^m}-1=q(a^{2^n}+1)$. Therefore,

$$\left(a^{2^m}-1\right)=q(a^{2^n}+1)-2.$$

By the Euclidean Algorithm,

$$\gcd(a^{2^m}-1, a^{2^n}+1)=\gcd(a^{2^n}+1,-2)=\begin{cases}1 \text{ if a is even}\\ 2 \text{ if a is odd}\end{cases}.$$

1.29 Assume for the sake of contradiction that $2^b-1 \mid 2^a+1$. We obviously have $a>b$, so write $a=bq+r$ using the division algorithm. We must have $\gcd(2^b-1, 2^a+1)=2^b-1$. We then have

$$\begin{aligned}\gcd(2^b-1, 2^a+1) &= \gcd(2^b-1, 2^a+1+2^b-1)\\ &= \gcd(2^b-1, 2^b\left(2^{a-b}+1\right)=\gcd(2^b-1, 2^{a-b}+1).\end{aligned}$$

Repeating this process, we arrive at

$$\gcd(2^b-1, 2^a+1)=\gcd(2^b-1, 2^{a-qb}+1)=\gcd(2^b-1, 2^r+1).$$

Since $r<b$, we have $2^r+1 \le 2^{b-1}+1<2^b-1$ for $a,b>2$.

1.30 [Outline] Use induction and the factorization $x^6-x^5+x^4-x^3+x^2-x+1=(x+1)^6-7x(x^2+x+1)^2$.

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