Departemen Matematika

Ujian Tengah Semester

Fungsi Khusus

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Soal

1 Hitunglah fungsi Gamma:

(a) $\Gamma(-2.3)$.

(b) $\int\limits_0^\infty x^5 e^{-4x}\, \mathrm{d}x$.

2 Hitunglah fungsi Beta:

(a) $\int\limits_0^{\pi/2} \sqrt{\cot(x)}\, \mathrm{d}x$.

(b) $\int\limits_2^5 \frac{\mathrm{d}x}{\sqrt{(x-2)(5-x)}}$.

3 Menggunakan relasi rekurensi fungsi Mittag-Leffler,

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)},$$

tentukan $\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2 E_{1,3}(z)\right)$.

4 (a) Menggunakan rumus Rodrigues, hitunglah polinom Legendre $P_3(x)$.

(b) Hitung $P_3^1(0.5)$ jika polinom Legendre associate derajat n dan orde m didefinisikan dengan

$$P_n^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_n(x).$$

Soal Nomor

1

Hitunglah fungsi Gamma:

(a). $\Gamma(-2.3)$.

(b). $\int\limits_0^\infty x^5 e^{-4x}\ \mathrm{d}x$.

Solusi:

(a) Menggunakan sifat $\Gamma(n+1) = n\Gamma(n) \iff \Gamma(n) = \frac{\Gamma(n+1)}{n}$, diperoleh

$$\Gamma(-2.3) = \frac{\Gamma(-1.3)}{(-2.3)} = \frac{\Gamma(-0.3)}{(-2.3)(-1.3)} = \frac{\Gamma(0.7)}{(-2.3)(-1.3)(-0.3)} = \frac{\Gamma(1.7)}{(-2.3)(-1.3)(-0.3)(0.7)}.$$

Berdasarkan $\Gamma(1.7) = 0.90864$, diperoleh $\Gamma(-2.3) = \boxed{-1.44711}$.

(b) Misalkan $u = 4x$, maka $\mathrm{d}u = 4\ \mathrm{d}x$. Diperoleh

$$\int\limits_0^\infty x^5 e^{-4x}\ \mathrm{d}x = \int\limits_0^\infty \left(\frac{u}{4}\right)^5 e^{-u}\ \frac{\mathrm{d}u}{4} = \frac{1}{4^6}\int\limits_0^\infty u^5 e^{-u}\ \mathrm{d}u = \frac{\Gamma(6)}{4^6} = \frac{5!}{4^6} = \boxed{\frac{30}{4^5}}.$$

SOAL NOMOR

2

Hitunglah fungsi Beta:

(a). $\displaystyle\int_0^{\pi/2} \sqrt{\cot(x)}\, \mathrm{d}x.$

(b). $\displaystyle\int_2^{5} \frac{\mathrm{d}x}{\sqrt{(x-2)(5-x)}}.$

Solusi:

Akan digunakan fakta $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(a). Perhatikan bahwa

$$\int_0^{\pi/2} \sqrt{\cot(x)}\, \mathrm{d}x = \int_0^{\pi/2} \Big(\cos(x)\Big)^{1/2} \Big(\sin(x)\Big)^{-1/2} \mathrm{d}x.$$

Perhatikan bahwa

$$\int_0^{\pi/2} \Big(\cos(x)\Big)^{2m-1} \Big(\sin(x)\Big)^{2n-1} \mathrm{d}x = \frac{1}{2}B(m,n).$$

Dengan memerhatikan nilai m dan n yang memenuhi $2m-1 = \dfrac{1}{2}$ dan $2n-1 = -\dfrac{1}{2}$, diperoleh $m = \dfrac{3}{4}$ dan $n = \dfrac{1}{4}$. Jadi,

$$\begin{aligned}\int_0^{\pi/2} \Big(\cos(x)\Big)^{1/2} \Big(\sin(x)\Big)^{-1/2} \mathrm{d}x &= \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{4}\right) \\ &= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} \\ &= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}\end{aligned}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2}$$

mengingat $\Gamma(1) = 1$. Menurut Euler Reflection Formula, yaitu $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ untuk $z \in \mathbb{C}$ dengan $\text{Re}(z) > 0$, dapat diperoleh bahwa untuk $z = \dfrac{1}{4}$ berlaku

$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{1}{2}\sqrt{2}} = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}.$$

Jadi, $\displaystyle\int_0^{\pi/2} \sqrt{\cot(x)}\, dx = \frac{1}{2}\cdot\pi\sqrt{2} = \boxed{\frac{\pi}{2}\sqrt{2}}$.

(b). Misalkan $x = 3u + 2$, maka $\mathrm{d}x = 3\ \mathrm{d}u$ dan diperoleh

$$\int_2^5 \frac{\mathrm{d}x}{\sqrt{(x-2)(5-x)}} = \int_0^1 \frac{3\ \mathrm{d}u}{\sqrt{3u(3-3u)}} = \int_0^1 \frac{\mathrm{d}u}{\sqrt{u(1-u)}} = \int_0^1 u^{1/2}(1-u)^{1/2}\ \mathrm{d}u.$$

Tinjau fakta bahwa $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\ \mathrm{d}x = B(m,n)$. Dengan memerhatikan nilai m dan n yang memenuhi $m - 1 = \dfrac{1}{2}$ dan $n - 1 = \dfrac{1}{2}$, diperoleh $m = n = \dfrac{3}{2}$. Ini berarti

$$\int_0^1 u^{1/2}(1-u)^{1/2}\ \mathrm{d}u = B\left(\frac{3}{2},\frac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} = \frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{\Gamma(3)} = \frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2!} = \frac{\frac{1}{4}\pi}{2} = \boxed{\frac{\pi}{8}}$$

mengingat $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.

SOAL NOMOR

3

Menggunakan relasi rekurensi fungsi Mittag-Leffler,

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)},$$

tentukan $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(z^2E_{1,3}(z)\right)$.

Solusi:

Perhatikan bahwa

$$E_{1,2}(z) = zE_{1,3}(z) + \frac{1}{\Gamma(2)} = zE_{1,3}(z) + 1 \implies z^2E_{1,3}(z) = zE_{1,2}(z) - z.$$

Diperoleh pula

$$E_{1,1}(z) = zE_{1,2}(z) + \frac{1}{\Gamma(1)} = zE_{1,2}(z) + 1 \implies zE_{1,2}(z) = E_{1,1}(z) - 1.$$

Dari sini diperoleh

$$z^2E_{1,3}(z) = zE_{1,2}(z) - z = E_{1,1}(z) - z - 1.$$

Perhatikan bahwa

$$E_{1,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z.$$

Dari sini diperoleh $z^2E_{1,3}(z) = e^z - z - 1$ yang memberikan

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2E_{1,3}(z)\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(e^z - z - 1\right) = \boxed{e^z - 1}.$$

Soal Nomor

4

(a) Menggunakan rumus Rodrigues, hitunglah polinom Legendre $P_3(x)$.

(b) Hitung $P_3^1(0.5)$ jika polinom Legendre associate derajat n dan orde m didefinisikan dengan

$$P_n^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_n(x).$$

Solusi:

(a) Akan digunakan $P_n(x) = \dfrac{1}{2^n n!} \dfrac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^2 - 1\right)^n$ sehingga diperoleh $P_3(x) = \dfrac{1}{8 \cdot 6} \dfrac{\mathrm{d}^3}{\mathrm{d}x^3} \left(x^2 - 1\right)^3$. Maka

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} \left(x^6 - 3x^4 + 3x^2 + 1\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(6x^5 - 12x^3 + 6x\right) = \frac{\mathrm{d}}{\mathrm{d}x} \left(30x^4 - 36x^2 + 6\right) = 120x^3 - 72x.$$

Ini berarti $P_3(x) = \dfrac{1}{48} \left(120x^3 - 72x\right) = \boxed{\dfrac{5x^3 - 3x}{2}}$.

(b) Diperoleh $P_3^1(x) = (-1)^1 \left(1 - x^2\right)^{1/2} \dfrac{\mathrm{d}}{\mathrm{d}x} \left(\dfrac{5x^3 - 3x}{2}\right)$ yang memberikan

$$P_3^1(x) = -\left(1 - x^2\right)^{1/2} \cdot \frac{15x^2 - 3}{2} = \frac{3}{2} \left(1 - 5x^2\right) \left(1 - x^2\right)^{1/2}.$$

Diperoleh pula

$$P_3^1(0.5) = \frac{3}{2} \left(1 - \frac{5}{4}\right) \sqrt{1 - \frac{1}{4}} = \frac{3}{2} \left(-\frac{1}{4}\right) \cdot \frac{\sqrt{3}}{2} = \boxed{-\frac{3\sqrt{3}}{16}}.$$