Euclidean Geometry in Mathematical Olympiad

Problems and Solutions
Chapter 3: Lengths and Ratios

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1 Soal

1. Let ABC be a triangle with contact triangle DEF. Prove that $\overline{AD}, \overline{BE}, \overline{CF}$ concur. The point of concurrency is the **Gergonne point** of triangle ABC.

2. *In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.*

3. Let $\overline{AD}, \overline{BE}, \overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that
$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

4. Let $ABCDE$ be a convex pentagon such that
$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$
Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$.

   Shortlist International Mathematical Olympiad 2006/Geometry 3

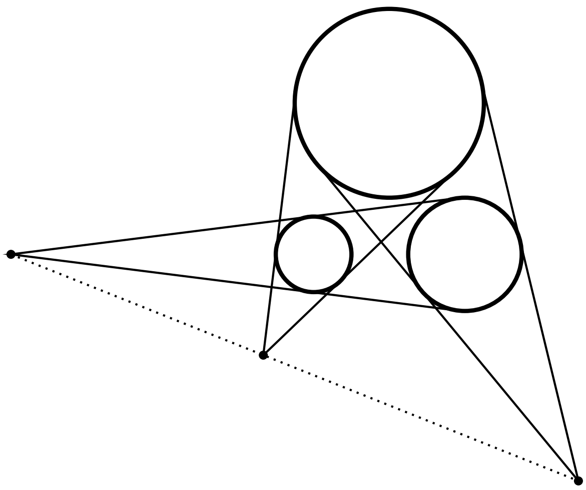
5. Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC.

   Bay Area Mathematical Olympiad 2013/Problem 3

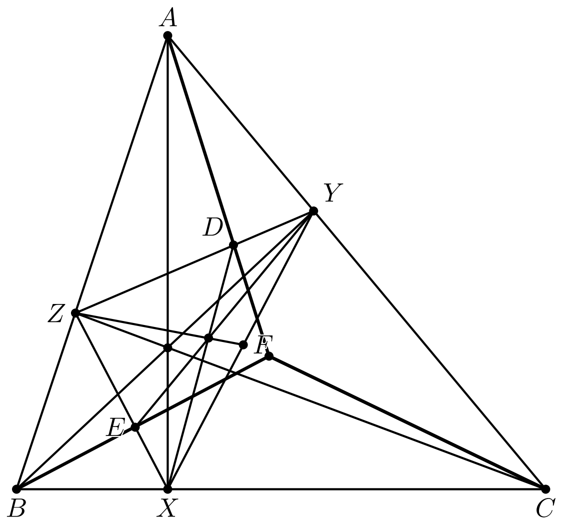
6. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

   USA Mathematical Olympiad 2003/Problem 4

7. **(Monge's Theorem).** *Consider disjoint circles $\omega_1, \omega_2, \omega_3$ in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.*



8. **(Cevian Nest).** *Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be concurrent cevians of ABC. Let $\overline{XD}, \overline{YE}, \overline{ZF}$ be concurrent cevians in triangle XYZ. Prove that rays $\overline{AD}, \overline{BE}, \overline{CF}$ concur.*



9. Let ABC be an acute triangle and suppose X is a point on (ABC) with $\overline{AX} \parallel \overline{BC}$ and $X \neq A$. Denote by G the centroid of triangle ABC, and by K the foot of the altitude from A to $\overline{BC}$. Prove that K, G, X are collinear.

10. Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are concyclic.

USA Mathematical Olympiad 1993/Problem 2

11. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.

European Girls' Mathematical Olympiad 2013/Problem 1

12. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.

Asian Pacific Mathematical Olympiad 2004/Problem 2

13. Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1 and C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

Shortlist International Mathematical Olympiad 2001/Geometry 1

14. Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

USA TST Selection Test 2011/Problem 4

15. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that XT is perpendicular to AX. Let M denote the midpoint of chord ST. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

USA Mathematical Olympiad 2015/Problem 2

2 Soal dan Solusi

1. Let ABC be a triangle with contact triangle DEF. Prove that $\overline{AD}, \overline{BE}, \overline{CF}$ concur. The point of concurrency is the **Gergonne point** of triangle ABC.

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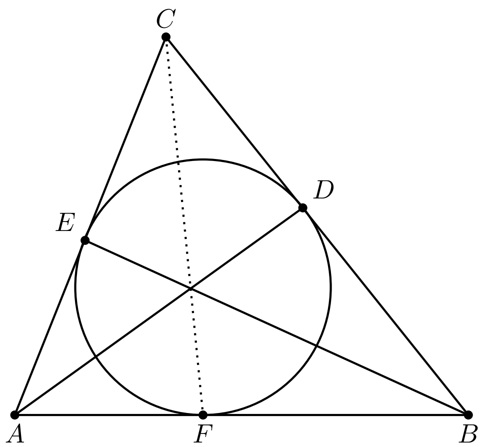

Perhatikan bahwa

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{s-a}{s-b} \cdot \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} = 1.$$

sehingga menurut **Teorema Ceva** berlaku AD, BE, CF konkuren.

2. *In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.*

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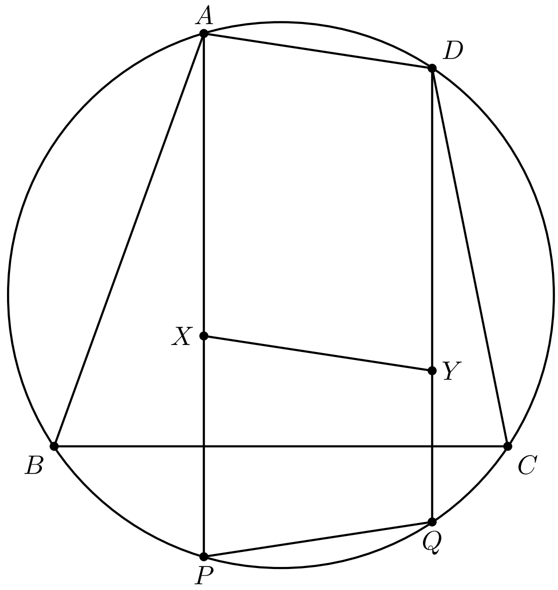

Misalkan P dan Q berturut-turut merupakan pencerminan titik X dan Y terhadap BC. Dari *Chapter 1*, kita punya titik P dan Q berada di $(ABCD)$. Selain itu, kita punya juga panjang $XY = PQ$. Karena $AP \perp BC$ dan $DQ \perp BC$, kita peroleh bahwa $AP \parallel DQ$ dan diperoleh $XYQP$ trapesium sama kaki. Karena $APQD$ siklis, maka

$$\measuredangle APQ = -\measuredangle PQD = \measuredangle PAD \implies \measuredangle APQ = \measuredangle PAD,$$

maka $APQD$ trapesium sama kaki. Kita punya

$$\measuredangle XAD = \measuredangle PAD = -\measuredangle APQ = -\measuredangle XPQ = \measuredangle PXY \implies \measuredangle XAD = \measuredangle PXY.$$

Maka $AD \parallel XY$. Karena juga $AX \parallel DY$, kita peroleh $AXYD$ jajargenjang.

3. Let $\overline{AD}, \overline{BE}, \overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

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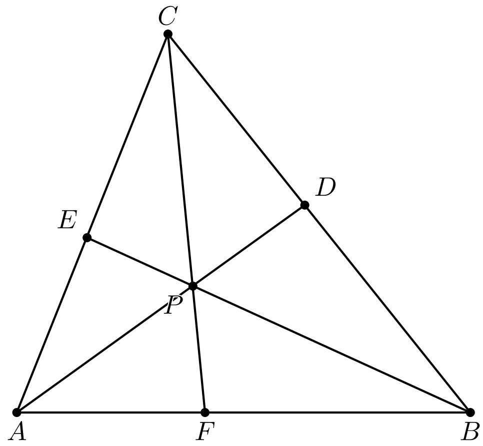

Perhatikan bahwa

$$\frac{[PDC]}{[ADC]} = \frac{PD}{AD} = \frac{[PDB]}{[ADB]} \implies \frac{PD}{AD} = \frac{[PDC]}{[ADC]} = \frac{[PDB]}{[ADB]} = \frac{[PDC]+[PDB]}{[ADC]+[ADB]} = \frac{[PBC]}{[ABC]}.$$

Maka $\frac{PD}{AD} = \frac{[PBC]}{[ABC]}$. Dengan cara yang sama, kita punya

$$\frac{PE}{BE} = \frac{[PAC]}{[ABC]} \quad \text{dan} \quad \frac{PF}{CF} = \frac{[PAB]}{[ABC]}.$$

Kita punya

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = \frac{[PBC]+[PAC]+[PAB]}{[ABC]} = \frac{[ABC]}{[ABC]} = 1$$

seperti yang ingin dibuktikan.

4. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$.

Shortlist International Mathematical Olympiad 2006/Geometry 3

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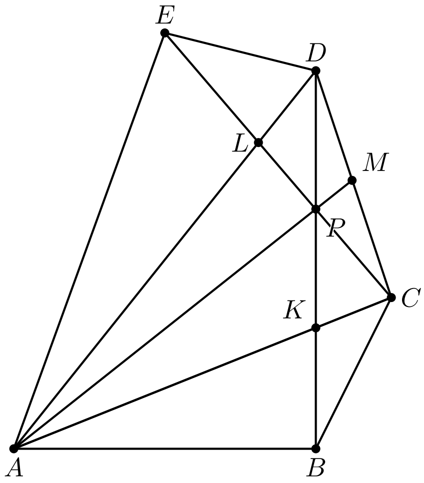
Misalkan $AP \cap \overline{CD} = M$, $BD \cap AC = K$, dan $CE \cap AD = L$.



Dari hubungan sudut-sudut, kita punya $\triangle ABC \sim \triangle ACD \sim \triangle ADE$. Dari hubungan tersebut, diperoleh $ABCD \sim ACDE$. Maka berlaku $\frac{AK}{KC} = \frac{AL}{LD} \iff \frac{AK}{KC} \cdot \frac{DL}{LA} = 1$. Karena AM, DK, CL konkuren, dari **Teorema Ceva**, berlaku

$$1 = \frac{AK}{KC} \cdot \frac{CM}{MD} \cdot \frac{DL}{LA} = \frac{CM}{MD} \implies CM = MD$$

seperti yang ingin dibuktikan.

5. Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC.

Bay Area Mathematical Olympiad 2013/Problem 3

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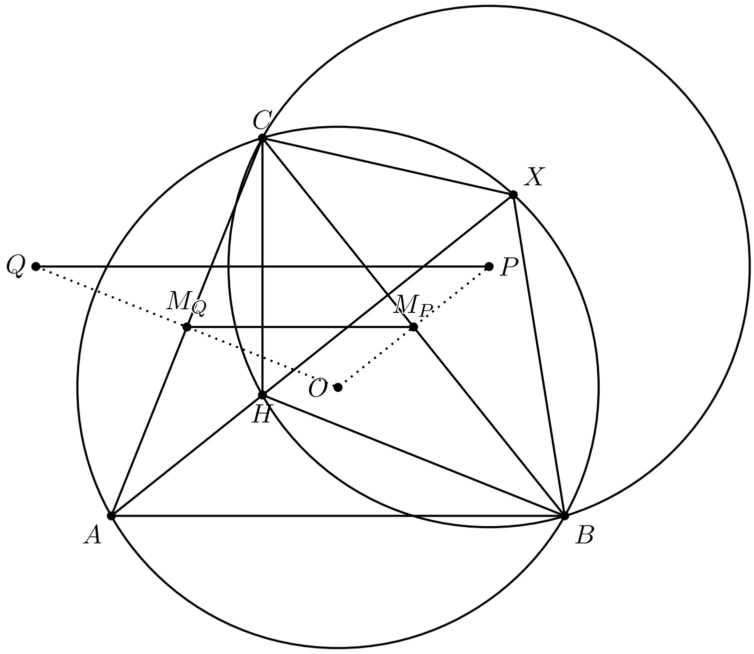
Misalkan P, Q, dan R berturut-turut merupakan pusat $(HBC), (HCA)$, dan (HAB). Misalkan pula O pusat (ABC). Misalkan X adalah pencerminan titik H terhadap $\overline{BC}$ di mana H adalah titik tinggi $\triangle ABC$.



Dari *Chapter 1*, kita punya $X \in (ABC)$. Tinjau bahwa $\triangle BHC \cong \triangle BXC$, sehingga $(BHC) \cong (BXC)$. Selain itu, karena X adalah bayangan pencerminan titik H terhadap $\overline{BC}$, maka (BHC) adalah bayangan pencerminan (ABC) terhadap $\overline{BC}$. Kita peroleh juga bahwa titik P merupakan bayangan pencerminan titik O terhadap $\overline{BC}$. Maka $OP \perp BC$ yang berarti OP melalui titik tengah BC, misalkan M_P. Secara analog, kita peroleh OQ melalui titik tengah $\overline{CA}$, misalkan M_Q dan OR melalui titik tengah AB, misalkan M_R. Dari **Midpoint Theorem**, kita peroleh $AB \parallel M_L M_P \parallel PQ \implies AB \parallel PQ$. Perhatikan bahwa

$$h(O, 2) : (M_P, M_Q, M_R) \mapsto (P, Q, R) \iff h(O, 2) : \triangle M_P M_Q M_R \mapsto \triangle PQR.$$

Kita punya juga $2M_P M_Q = PQ$. Selain itu, tinjau bahwa

$$h(C, 2) : (M_P, M_Q) \mapsto (B, A) \iff AB = 2M_P M_Q = PQ \implies AB = PQ.$$

Secara analog, kita peroleh $QR = BC$ dan $RP = AC$ yang mana berakibat $\triangle ABC \cong \triangle PQR$ seperti yang ingin dibuktikan.

6. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

USA Mathematical Olympiad 2003/Problem 4

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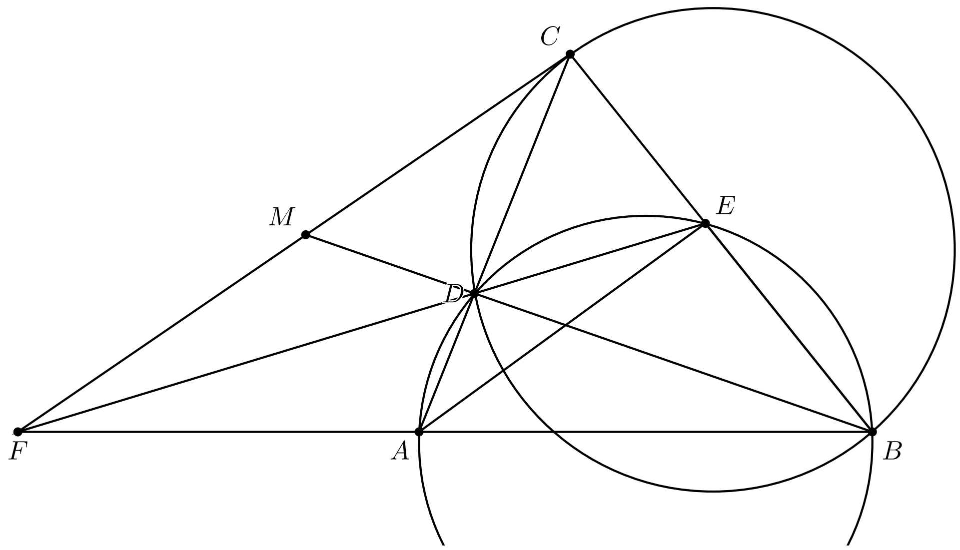


Pembuktian dari kiri ke kanan, yaitu jika panjang $MF = MC$. Karena BM, CA, FE konkuren, dari **Teorema Ceva** berlaku

$$1 = \frac{FA}{AB} \cdot \frac{BE}{EC} \cdot \frac{CM}{MF} = \frac{FA}{AB} \cdot \frac{BE}{EC} \implies \frac{FA}{AB} = \frac{EC}{BE} \iff \frac{FB}{AB} = \frac{CB}{BE}.$$

Karena $\angle ABE = \angle FBC$, maka $\triangle ABE \sim \triangle FBC$. Kita punya $AE \parallel FC$. Karena $ABED$ siklis, maka

$$\measuredangle MCD = \measuredangle FCA = \measuredangle EAC = \measuredangle EAD = \measuredangle EBD \implies \measuredangle MCD = \measuredangle EBD.$$

Dari **Alternate Segment Theorem**, maka MC garis singgung (BDC). Dari **Power Of a Point** berlaku $MB \cdot MD = MC^2$.

Pembuktian dari kanan ke kiri, yaitu jika $MB \cdot MD = MC^2 \iff MC$ garis singgung (BDC). Dari **Alternate Segment Theorem** berlaku

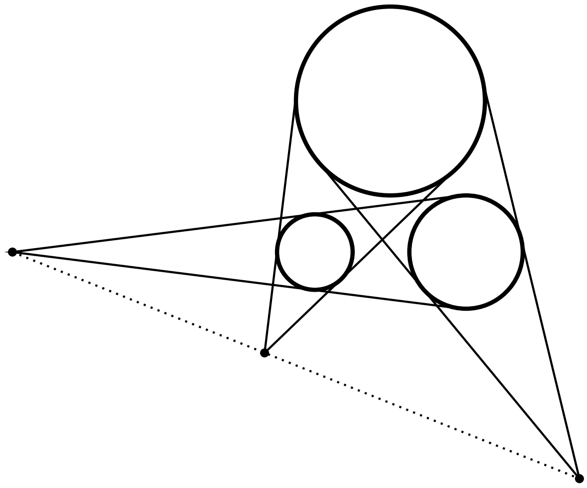
$$\measuredangle MCD = \measuredangle EBD \iff \measuredangle FCA = \measuredangle EAD = \measuredangle EAC \implies \measuredangle FCA = \measuredangle EAC.$$

Maka $FC \parallel AE$ sehingga diperoleh $\angle AEB = \angle FCB$ dan $\angle ABE = \angle FBC$. Kita punya $\triangle ABE \sim \triangle FBC$. Kita punya $\frac{FB}{AB} = \frac{CB}{BE} \iff \frac{FA}{AB} = \frac{EC}{BE} \iff \frac{FA}{AB} \cdot \frac{BE}{EC} = 1$. Karena BM, EF, AC konkuren, dari **Teorema Ceva** berlaku

$$1 = \frac{CM}{MF} \cdot \frac{FA}{AB} \cdot \frac{BE}{EC} = \frac{CM}{MF} \implies CM = MF.$$

Maka terbukti $MF = MC \iff MB \cdot MD = MC^2$.

7. **(Monge's Theorem).** *Consider disjoint circles $\omega_1, \omega_2, \omega_3$ in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.*



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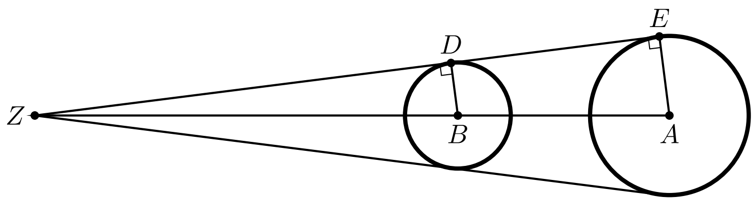
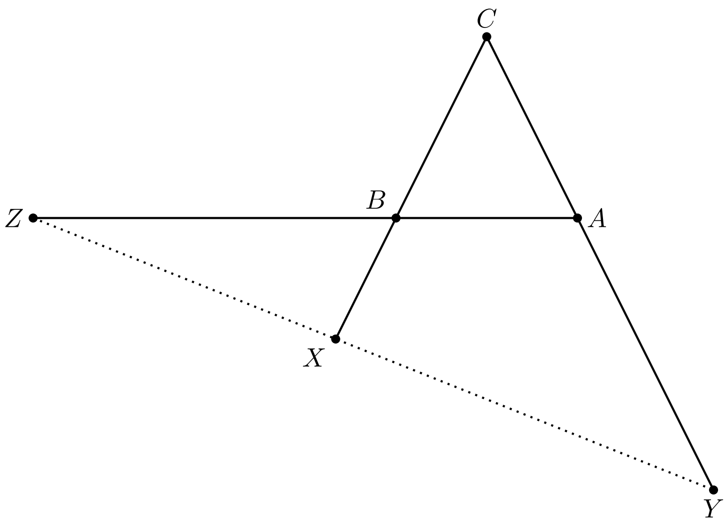
Misalkan r_1, r_2, r_3 berturut-turut menyatakan panjang jari-jari lingkaran yang berpusat di A, B, dan C. Misalkan pula X, Y, Z berturut-turut perpotongan garis singgung lingkaran B dan lingkaran C, lingkaran A dan lingkaran C, lingkaran A dan lingkaran B. Perhatikan dua lingkaran berikut yang berjari-jari r_1 dan r_2.



Karena $\angle BDZ = \angle AEZ$ dan $\angle DZB = \angle EZA$, maka $\triangle BDZ \sim \triangle AEZ$. Kita punya $\frac{AZ}{BZ} = \frac{AE}{BD} \iff \frac{AZ}{BZ} = \frac{r_1}{r_2}$. Secara analog, kita punya $\frac{BX}{CX} = \frac{r_2}{r_3}$ dan $\frac{CY}{AY} = \frac{r_3}{r_1}$. Konstruksi ulang gambar seperti berikut.

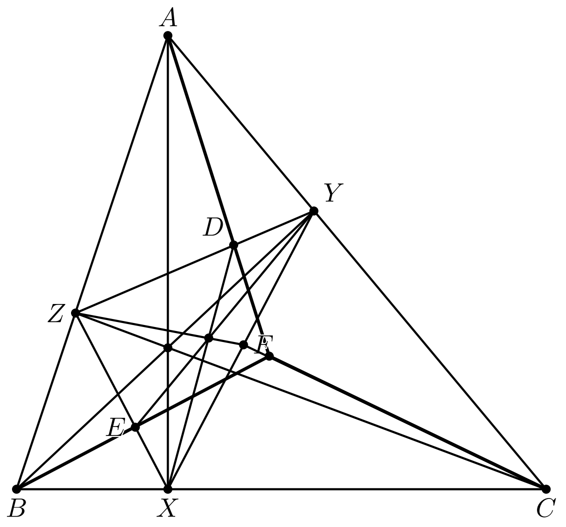


Kita punya

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{r_1}{r_2} \cdot \frac{r_2}{r_3} \cdot \frac{r_3}{r_1} = 1,$$

maka menurut **Teorema Menelaus** berlaku X, Y, dan Z kolinear seperti yang ingin dibuktikan.

8. **(Cevian Nest).** *Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be concurrent cevians of ABC. Let $\overline{XD}, \overline{YE}, \overline{ZF}$ be concurrent cevians in triangle XYZ. Prove that rays $\overline{AD}, \overline{BE}, \overline{CF}$ concur.*



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Karena XD, YE, ZF konkuren, menurut **Teorema Ceva** berlaku

$$\frac{YD}{DZ} \cdot \frac{ZE}{EX} \cdot \frac{XF}{FY} = 1. \qquad (*)$$

Dari aturan sinus $\triangle ADY$ dan $\triangle ADZ$, kita punya

$$\frac{YD}{DZ} = \frac{\frac{AY}{\sin \angle ADY} \cdot \sin \angle DAY}{\frac{AZ}{\sin \angle ADZ} \cdot \sin \angle ZAD} = \frac{AY}{AZ} \cdot \frac{\sin \angle ADZ}{\sin \angle ADY} \cdot \frac{\sin \angle DAY}{\sin \angle ZAD}.$$

Karena $\angle ADY = \sin(180^\circ - \angle ADZ) = \sin \angle ADZ$, kita punya

$$\frac{YD}{DZ} = \frac{AY}{AZ} \cdot \frac{\sin \angle DAY}{\sin \angle ZAD} = \frac{AY}{AZ} \cdot \frac{\sin \angle CAD}{\sin \angle BAD}.$$

Secara analog, kita punya

$$\frac{ZE}{EX} = \frac{BZ}{BX} \cdot \frac{\sin \angle ABE}{\sin \angle CBE} \quad \text{dan} \quad \frac{XF}{FY} = \frac{XC}{YC} \cdot \frac{\sin \angle BCF}{\sin \angle ACF}.$$

Subtitusi ke $(*)$, kita punya

$$1 = \frac{AY}{AZ} \cdot \frac{\sin \angle CAD}{\sin \angle BAD} \cdot \frac{BZ}{BX} \cdot \frac{\sin \angle ABE}{\sin \angle CBE} \cdot \frac{CX}{CY} \cdot \frac{\sin \angle BCF}{\sin \angle ACF}$$
$$1 = \left(\frac{AY}{YC} \cdot \frac{CX}{XB} \cdot \frac{BZ}{ZA} \right) \left(\frac{\sin \angle CAD}{\sin \angle BAD} \cdot \frac{\sin \angle ABE}{\sin \angle CBE} \cdot \frac{\sin \angle BCF}{\sin \angle ACF} \right).$$

Karena AX, BY, CZ konkuren, dari **Teorema Ceva** berlaku

$$1 = \frac{AY}{YC} \cdot \frac{CX}{XB} \cdot \frac{BZ}{ZA}.$$

Sehingga kita punya

$$1 = \frac{\sin \angle CAD}{\sin \angle BAD} \cdot \frac{\sin \angle ABE}{\sin \angle CBE} \cdot \frac{\sin \angle BCF}{\sin \angle ACF}$$

yang berarti menurut **Teorema Ceva** berlaku AD, BE, CF konkuren.

9. Let ABC be an acute triangle and suppose X is a point on (ABC) with $\overline{AX} \parallel \overline{BC}$ and $X \neq A$. Denote by G the centroid of triangle ABC, and by K the foot of the altitude from A to $\overline{BC}$. Prove that K, G, X are collinear.

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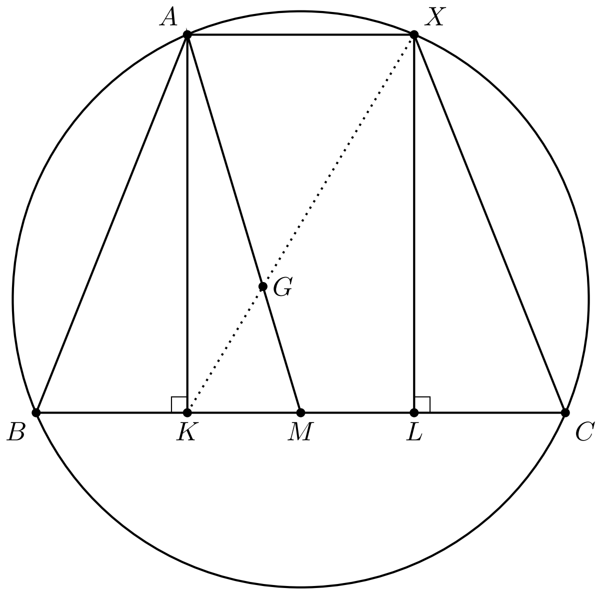
Misalkan $AG \cap BC = M$ dan L kaki tinggi dari X ke BC.



Karena $AX \parallel BC$ dan $ABCX$ segiempat tali busur, maka $\measuredangle ABC = \measuredangle BAX = \measuredangle BCX \implies \measuredangle ABC = \measuredangle BCX$. Maka $ABCX$ trapesium sama kaki sehingga panjang $AB = XC$. Dari **Teorema Pythagoras**, kita punya

$$BK = \sqrt{AB^2 - AK^2} = \sqrt{XC^2 - XL^2} = LC \implies BK = CL.$$

Karena panjang $BM = MC$, maka panjang $KM = ML = \frac{1}{2}AX \implies \frac{KM}{AX} = \frac{1}{2}$. Selain itu, kita juga tahu bahwa $\frac{MG}{GA} = \frac{1}{2}$ dan $\angle XAM = \angle AMK \implies \angle XAG = \angle KMG$. Karena juga $\frac{MG}{GA} = \frac{MK}{AX}$, kita simpulkan $\triangle KMG \sim \triangle XGA$. Maka berlaku $\angle KGM = \angle AGX \iff K, G, X$ kolinear.

10. Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are concyclic.

USA Mathematical Olympiad 1993/Problem 2

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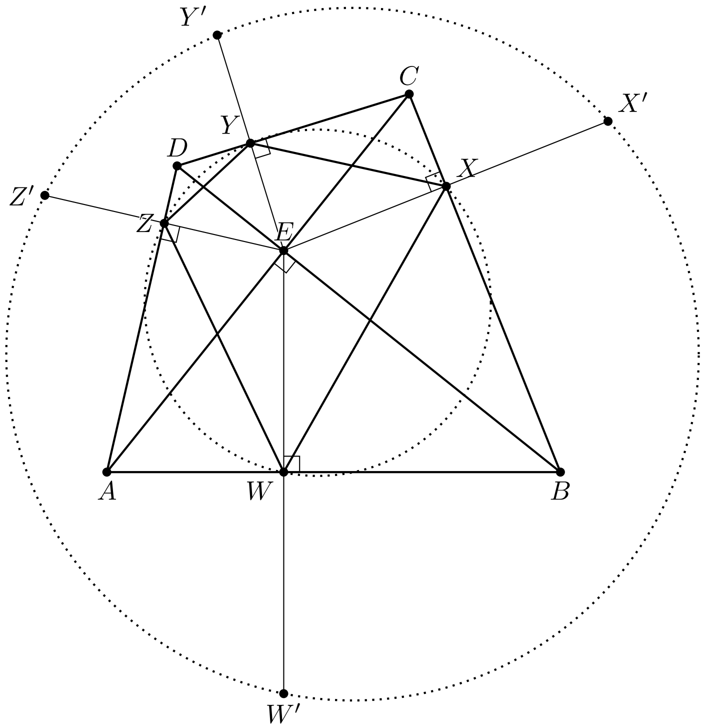
Misalkan W', X', Y', Z' berturut-turut adalah pencerminan titik E terhadap $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ dan $W = W'E \cap AB, X = X'E \cap BC, Y = Y'E \cap CD$, dan $Z = Z'E \cap DA$.



Perhatikan bahwa

$$h(E,2) : (W,X,Y,Z) \mapsto (W',X',Y',Z') \iff h(E,2) : WXYZ \mapsto W'X'Y'Z'.$$

Maka $W'X'Y'Z'$ siklis jika dan hanya jika $WXYZ$ siklis. Karena $\angle BWE + \angle BXE = 180^\circ$, maka $BWEX$ siklis. Secara analog, $CXEY, YEZD$, dan $AWEZ$ siklis. Kita punya

$$\angle EWX = \angle EBX = \angle EBC, \quad \angle EWZ = \angle EAZ = \angle EAD,$$
$$\angle EYX = \angle ECX = \angle ECB, \quad \angle EYZ = \angle EDZ = \angle EDA.$$

Kita punya

$$\begin{aligned}\angle ZWX + \angle ZYX &= \angle EBC + \angle EAD + \angle ECB + \angle EDA \\ &= (\angle EBC + \angle ECB) + (\angle EAD + \angle EDA) \\ &= 90^\circ + 90^\circ \\ &= 180^\circ.\end{aligned}$$

Maka $WXYZ$ siklis seperti yang ingin dibuktikan.

11. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.

European Girls' Mathematical Olympiad 2013/Problem 1

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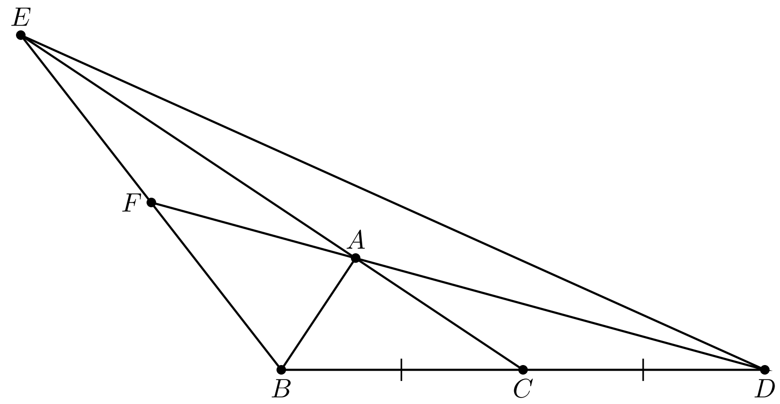
Misalkan $DA \cap BE = F$.



Karena panjang $CB = CD$ dan $EA : AC = 2 : 1$, maka A titik berat $\triangle BDE$. Maka panjang $EF = BF = n$ dan panjang $AD = 2n$. Karena $DA : AF = 2 : 1$, maka panjang $AF = n$. Karena panjang $AF = BF = FE$, maka F titik pusat lingkaran luar $\triangle ABE$. Karena BE diameter (ABE), maka $\angle BAE = 90^\circ \iff \angle BAC = 90^\circ$ seperti yang ingin dibuktikan.

12. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.

Asian Pacific Mathematical Olympiad 2004/Problem 2

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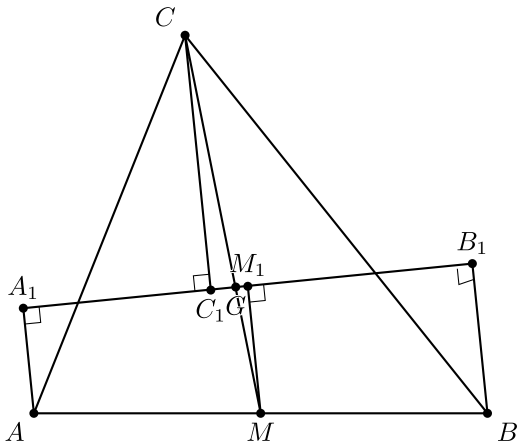
W.L.O.G. titik A dan B berada pada sisi yang sama terhadap garis OH. Misalkan M titik tengah $\overline{AB}$ sehingga kita peroleh $AM \cap OH = G$ di mana G titik berat $\triangle ABC$. Misalkan pula A_1 adalah hasil proyeksi titik A ke garis OH. Definisikan yang sama untuk B_1, C_1, dan M_1. Maka

$$[AOH] + [BOH] = [COH] \iff \frac{AA_1 \cdot OH}{2} + \frac{BB_1 \cdot OH}{2} = \frac{CC_1 \cdot OH}{2} \iff AA_1 + BB_1 = CC_1.$$



Kita punya $\triangle MM_1G \sim \triangle CC_1G$ dan diperoleh bahwa

$$\frac{CC_1}{MM_1} = \frac{CG}{MG} = 2 \iff CC_1 = 2MM_1.$$

Maka sekarang ekuivalen dengan membuktikan $AA_1 + BB_1 = 2MM_1$ yang mana jelas benar pada trapesium AA_1BB_1 mengingat M titik tengah $\overline{AB}$.

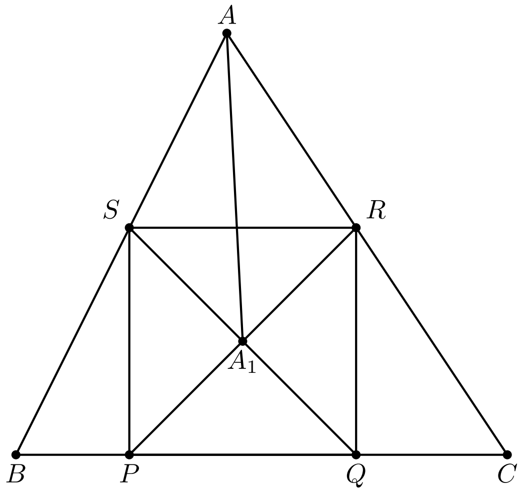
> **Remark.** Diberikan trapesium $ABCD$ di mana $AB \parallel CD$. Titik M_1 dan M_2 berturut-turut pada BC dan DA sehingga $MN \parallel AB$. Maka
> $$MN = \frac{AB \cdot CM + CD \cdot BM}{BC},$$
> yang mana jika panjang $CM = MB$ diperoleh bahwa $2MN = AB + CD$. Pembuktian diserahkan kepada pembaca.

13. Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1 and C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

Shortlist International Mathematical Olympiad 2001/Geometry 1

. . . .

Misalkan persegi $PQRS$ adalah pusat persegi yang berpusat di A_1 di mana P dan Q berada di $\overline{BC}$, R berada di $\overline{AC}$, dan S berada di $\overline{AB}$.



Tinjau $\angle RSP = \angle SPB$, maka $SR \parallel BC$. Kita punya $\angle ASR = \angle ABC = \angle B$ dan $\angle ARS = \angle ACB = \angle C$. Maka $\angle ASA_1 = 45^\circ + \angle B$ dan $\angle ARA_1 = 45^\circ + \angle C$. Dari aturan sinus $\triangle ASA_1$ dan $\triangle ARA_1$, diperoleh

$$\frac{\sin \angle BAA_1}{\sin \angle CAA_1} = \frac{\sin \angle SAA_1}{\sin \angle RAA_1} = \frac{\frac{SA_1}{AA_1} \cdot \sin \angle ASA_1}{\frac{RA_1}{AA_1} \cdot \sin \angle ARA_1} = \frac{\sin \angle ASA_1}{\sin \angle ARA_1} = \frac{\sin (45^\circ + B)}{\sin (45^\circ + C)}.$$

Dengan cara yang sama, diperoleh

$$\frac{\sin \angle CBB_1}{\sin \angle ABB_1} = \frac{\sin (45^\circ + C)}{\sin (45^\circ + B)} \quad \text{dan} \quad \frac{\sin \angle ACC_1}{\sin \angle BCC_1} = \frac{\sin (45^\circ + A)}{\sin (45^\circ + B)}.$$

Kita punya

$$\frac{\sin \angle BAA_1}{\sin \angle CAA_1} \cdot \frac{\sin \angle ACC_1}{\sin \angle BCC_1} \cdot \frac{\sin \angle CBB_1}{\sin \angle ABB_1} = \frac{\sin (45^\circ + B)}{\sin (45^\circ + C)} \cdot \frac{\sin (45^\circ + A)}{\sin (45^\circ + B)} \cdot \frac{\sin (45^\circ + C)}{\sin (45^\circ + A)} = 1,$$

sehingga menurut **Teorema Ceva** berlaku AA_1, BB_1, dan CC_1 konkuren.

14. Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

USA TST Selection Test 2011/Problem 4

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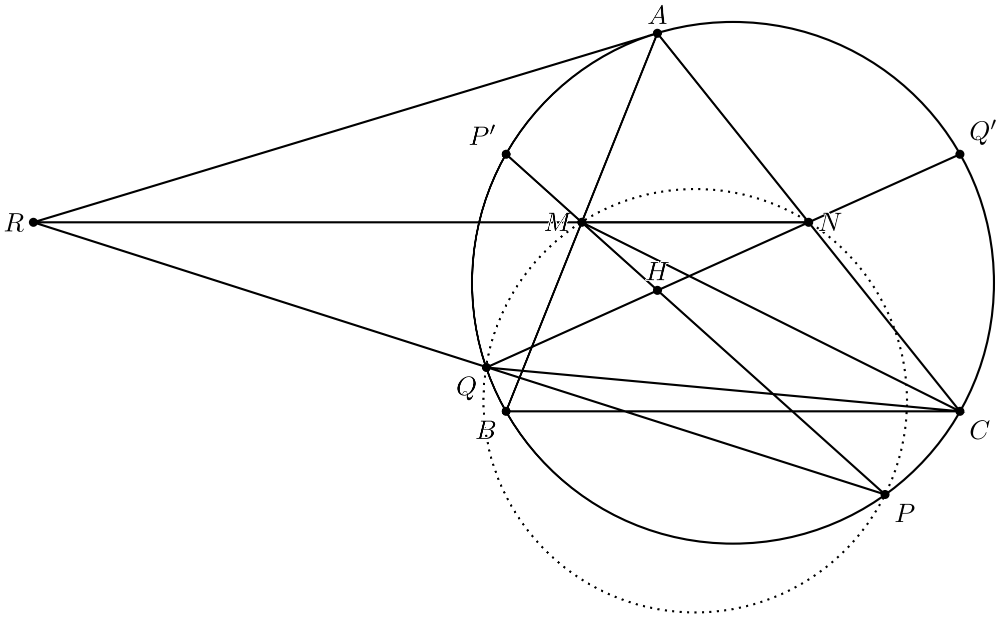
Misalkan PM dan QN berturut-turut memotong ω sekali lagi di titik P' dan Q'.



Klaim — $MNPQ$ siklis.

Dari **Teorema Power of Point**, maka $HP' \cdot HP = HQ' \cdot HQ$. Kita tahu bahwa $h(H,2) : M \mapsto P'$ dan $h(H,2) : N \mapsto Q'$ (lihat tentang nine point circle). Kita punya panjang $HM = \frac{1}{2}HP'$ dan $HN = \frac{1}{2}HQ'$. Maka

$$HP' \cdot HP = HQ' \cdot HQ \iff 2MH \cdot HP = 2HN \cdot HQ \iff MH \cdot HP = HN \cdot HQ,$$

maka $MNPQ$ siklis. Konstruksikan lingkaran (AMN). Tinjau bahwa PQ radical axis $(MNPQ)$ dan ω, sedangkan MN radical axis $(MNPQ)$ dan (AMN). Dari **Radical Axis Theorem**, maka AR radical axis (AMN) dan ω. Artinya, AR garis singgung dari ω dan (AMN) sehingga diperoleh bahwa $AR \perp OA$.

15. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that XT is perpendicular to AX. Let M denote the midpoint of chord ST . As X varies on segment $\overline{PQ}$, show that M moves along a circle.

USA Mathematical Olympiad 2015/Problem 2

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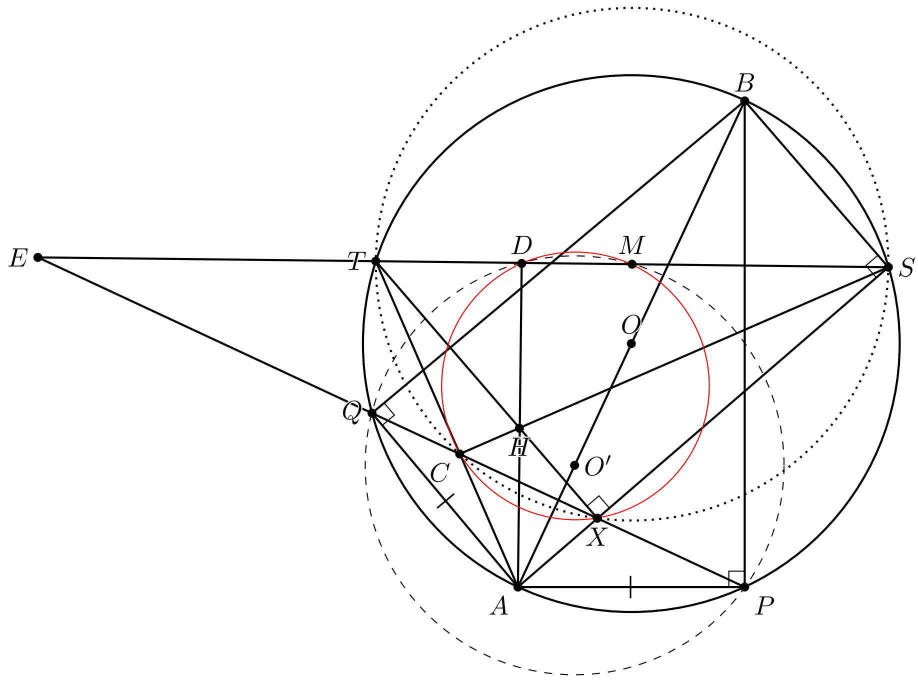
Misalkan pula C adalah perpotongan PQ dan AT, H adalah perpotongan TX dan SC, dan O pusat ω. Tinjau bahwa $AP = AQ$ sehingga $APBQ$ merupakan layang-layang dan kita peroleh $AB \perp PQ$. Misalkan O' adalah titik tengah $\overline{AO}$.



Klaim — $SC \perp AT$.

Tinjau bahwa

$$\measuredangle ACX = 90^\circ - \measuredangle BAC = 90^\circ - \measuredangle BAT = 90^\circ - \measuredangle BST = \measuredangle TSA = \measuredangle TSX \implies \measuredangle ACX = \measuredangle TSA$$

sehingga $SXCT$ siklis. Kita punya $\angle SCT = \angle SXT = 90^\circ$ dan klaim kita terbukti (selain itu hal ini menunjukkan PQ tidak mungkin sejajar ST). Maka H adalah titik tinggi dari $\triangle AST$ dan misalkan E perpotongan PQ dan ST. Misalkan AH memotong BC di D, maka $AD \perp BC$.

Klaim — M, D, P, Q siklis dengan lingkaran luarnya berpusat di titik tengah $\overline{AO}$.

Kita punya C, X, M, D siklis dengan lingkaran luarnya adalah nine point circle. Tinjau pula $TSXC$ dan $TSPQ$ siklis, dari **Teorema Power of Point**, kita punya

$$EQ \cdot EP = ET \cdot ES = EC \cdot EX = ED \cdot EM \implies EQ \cdot EP = ED \cdot EM.$$

Maka D, M, P, Q siklis dan misalkan lingkaran luarnya adalah Ω. Misalkan M' adalah hasil proyeksi O' pada $\overline{ST}$. Kita punya $OM \parallel O'M' \parallel AD$ mengingat $OM \perp ST$. Maka $MODA$ merupakan trapesium karena O' titik tengah AO, maka M' adalah titik tengah MD. Maka $O'M'$ garis sumbu $\overline{MD}$. Selain itu, kita tahu bahwa AO' merupakan garis sumbu PQ mengingat $APBQ$ layang-layang. Kita peroleh bahwa O' titik pusat Ω. Kita tahu bahwa P, Q, dan O' tetap, maka Ω juga tetap sehingga M berada di Ω untuk sembarang posisi X di $\overline{PQ}$.