Euclidean Geometry in Mathematical Olympiad

Problems and Solutions

Chapter 2: Circles

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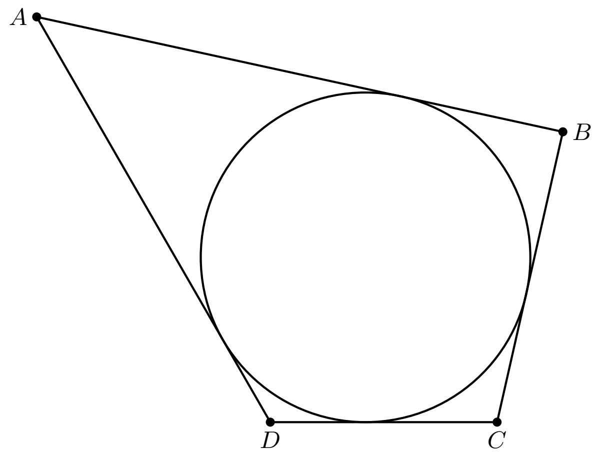
1 Soal

1. *Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.*

2. *Let $ABCD$ be a quadrilateral. If a circle can be incribed in it, prove that $AB + CD = BC + DA$.*



3. An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

   USA Mathematical Olympiad 1990/Problem 5

4. Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.

   (a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N.

   (b) Prove that no matter where one chooses the point M along segment AB, all lines MN will pass through some fixed point K in the plane.

   Bay Area Mathematical Olympiad 2012/Problem 4

5. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

   USA Junior Mathematical Olympiad 2012/Problem 1

6. Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.

   International Mathematical Olympiad 2008/Problem 1

7. Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$ respectively are concurrent.

   USA Mathematical Olympiad 1997/Problem 2

8. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter $\overline{BD}$ at B and N. Prove that the lines AM, DN, XY are concurrent.

   International Mathematical Olympiad 1995/Problem 1

9. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

   USA Mathematical Olympiad 1998/Problem 2

10. Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

    International Mathematical Olympiad 2000/Problem 1

11. Let $ABCD$ be a cyclic quadrilateral whose diagonals meet at P. Let W, X, Y, Z be the feet of P onto $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$, respectively. Show that $WX + YZ = XY + WZ$.

    Canada 1990/Problem 3

12. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

    International Mathematical Olympiad 2009/Problem 2

13. Let $\overline{AD}, \overline{BE}, \overline{CF}$ be the altitudes of a scalene triangle ABC with circumcenter O. Prove that $(AOD), (BOE)$, and (COF) intersect at point X other than O.

14. Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let $\omega, \omega_1, \omega_2$, and ω_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.

    (a) Prove that ω_1, ω_2, and ω_3 intersect in a common point.

    (b) Show that lines PD, QE, and RF are concurrent.

    Canada 2007/Problem 5

15. In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of $\overline{BC}$ and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MF}$, respectively, and let T be on line KL such that $\overline{TA} \parallel \overline{BC}$. Prove that $TA = TM$.

    Iran Team Selection Test 2011/Problem 1

2 Soal dan Solusi

1. *Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.*

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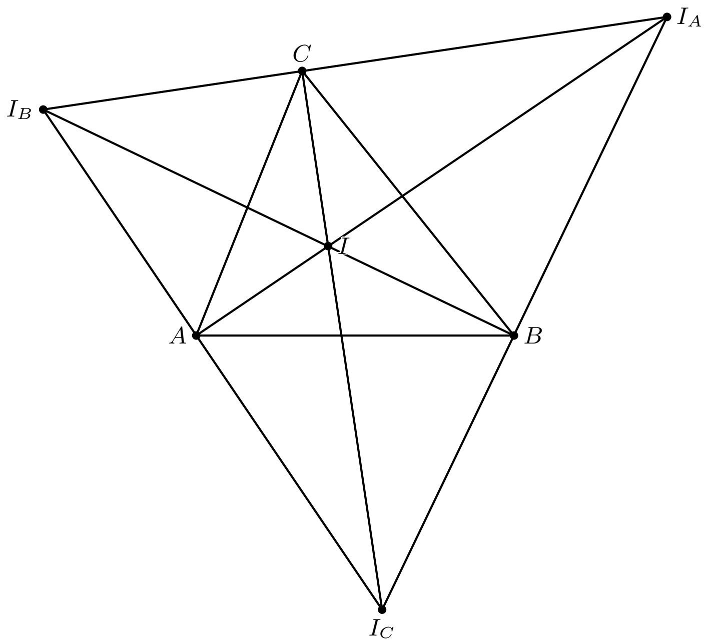
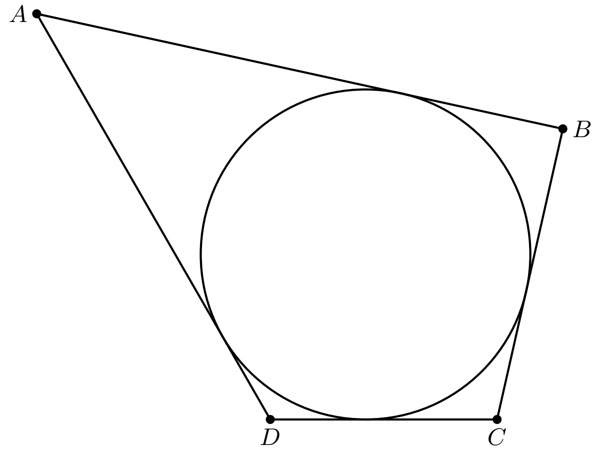
   

   Jelas bahwa I_A, C, I_B segaris. Begitu juga untuk I_B, A, I_C dan I_A, B, I_C. Dari **Incenter-Excenter Lemma**, kita punya II_A merupakan diameter $(BICI_A)$. Maka $\angle IBI_A = 90^\circ \implies \angle I_BBI_A = 90^\circ$. Maka $I_BB \perp I_CI_A$. Secara analog, $I_AA \perp I_BI_C$ dan $I_CC \perp I_AI_B$. Maka I merupakan titik tinggi $\triangle I_AI_BI_C$ dan $\triangle ABC$ sebagai segitiga pedal (*orthic triangle*) dari segitiga tersebut.

2. *Let $ABCD$ be a quadrilateral. If a circle can be incribed in it, prove that $AB + CD = BC + DA$.*



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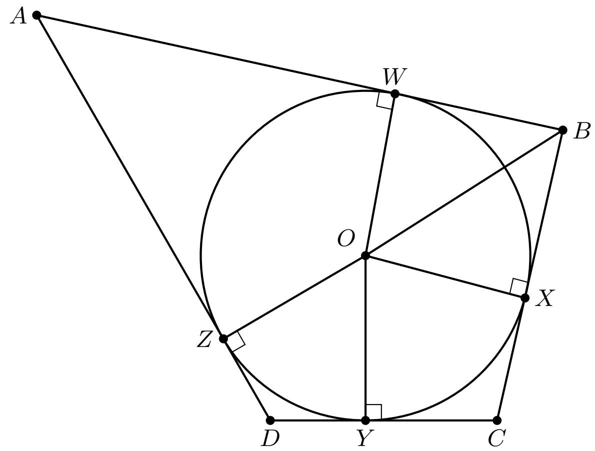
Misalkan lingkaran tersebut berpusat di O dan menyinggung $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ berturut-turut di titik W, X, Y, Z.



Perhatikan bahwa $\angle OXB = \angle OWB$, panjang $OB = OB$, dan panjang $OW = OX$, maka $\triangle OWB \cong \triangle OXB$. Maka panjang $BW = BX = b$. Secara analog, kita peroleh $AW = AZ = a, CX = CY = c$, dan $DY = DZ = d$. Maka kita bisa peroleh

$$AB + CD = a + b + c + d = BC + DA \implies AB + CD = BC + DA.$$

3. An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

USA Mathematical Olympiad 1990/Problem 5

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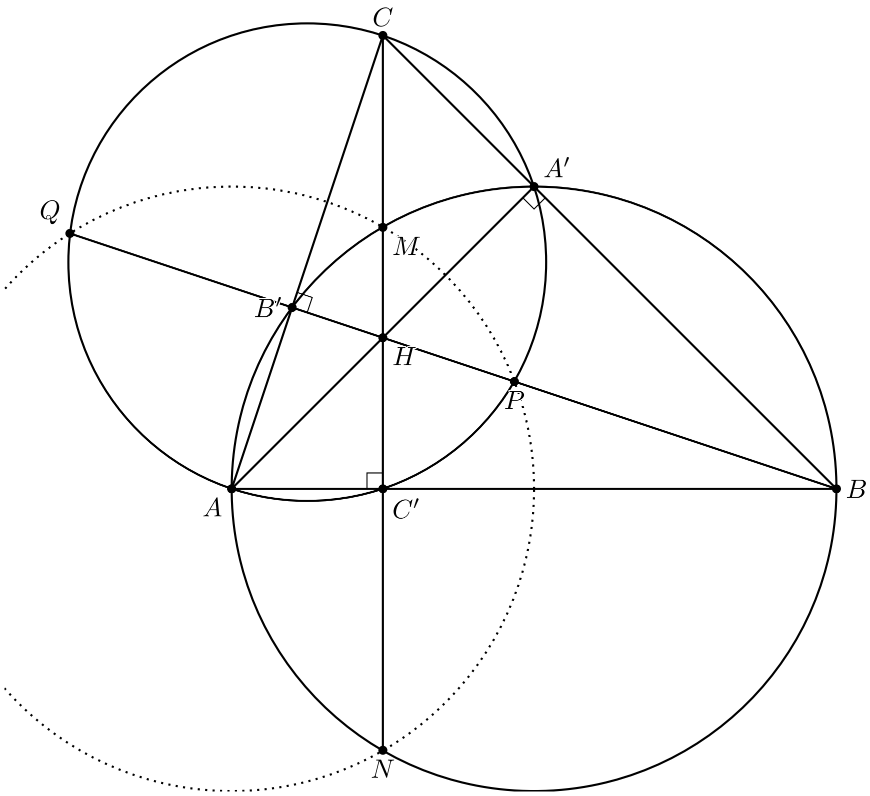
Misalkan $BB' \cap CC' = H$ dan $\overline{AA'}$ merupakan garis tinggi $\triangle ABC$ dari titik A.



Misalkan ω_1 dan ω_2 berturut-turut merupakan lingkaran berdiameter AB dan berdiameter AC. Dari ω_1, maka berlaku $MH \cdot HN = AH \cdot HA'$. Dari ω_2, maka berlaku $PH \cdot HQ = AH \cdot HA'$. Maka

$$MH \cdot HN = AH \cdot HA' = PH \cdot HQ \implies MH \cdot HN = PH \cdot HQ$$

yang berakibat M, N, P, Q siklis.

4. Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.
   (a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N.
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Bay Area Mathematical Olympiad 2012/Problem 4

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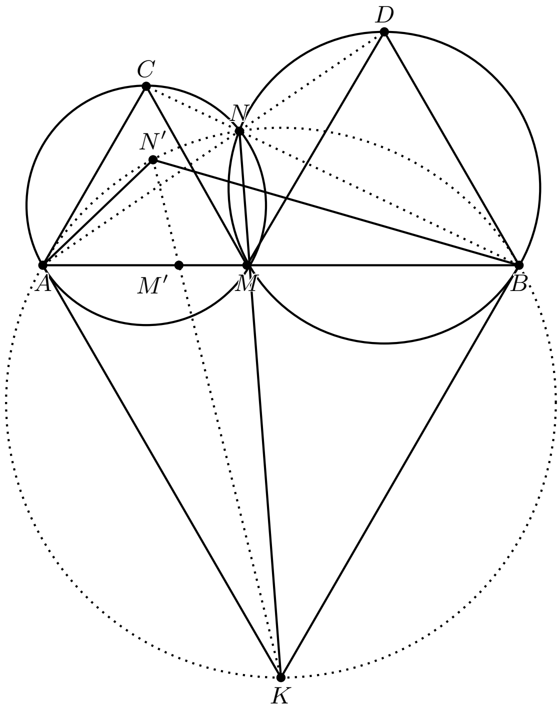

(a) Karena $AMNC$ siklis, maka $\angle ANM = \angle ACM = 60°$. Karena $BMND$ siklis, maka $\angle MND = 180° - \angle MBD = 120°$. Karena $\angle ANM + \angle MND = 180°$, maka A, N, D segaris. Secara analog, B, N, C segaris. Maka terbukti $\overline{AD}$ dan $\overline{BC}$ keduanya melalui N.

(b) Misalkan garis NM memotong (ANB) di titik K. Akan kita buktikan untuk semua garis NM untuk sembarang pemilihan M di $\overline{AB}$ akan melalui K. Ambil titik $M' \neq M$ di $\overline{AB}$ dan definisikan N' dengan cara yang sama. Akan kita buktikan A, N', N, B siklis. Tinjau $\angle ANB = 180° - \angle BND = 180° - \angle BMD = 120°$. Dengan cara yang sama, bisa diperoleh $\angle AN'B = 120° \implies \angle ANB = \angle AN'B$. Akibatnya, $AN'NB$ siklis. Karena $ANBK$ siklis, maka $\angle ABK = \angle ANK = \angle ANM = 60°$. Secara analog, kita peroleh juga $\angle BAK = 60°$. Maka $\angle KAB = \angle KBA \iff KA = KB$ yang mana K titik tengah busur AB. Jika $N'M'$ memotong $(AN'B)$ di K', secara analog bisa kita peroleh $K'A = K'B$ sehingga K' titik tengah busur AB. Maka $K = K'$ seperti yang ingin dibuktikan.

5. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

USA Junior Mathematical Olympiad 2012/Problem 1

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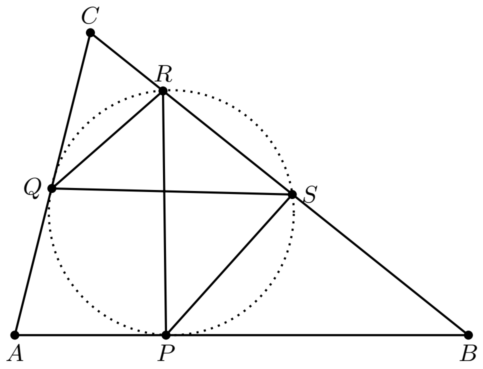

Misalkan $(PRS) = \omega_1$ dan $(QRS) = \omega_2$. Akan kita buktikan bahwa $\omega_1 = \omega_2$. Andaikan $\omega_1 \neq \omega_2$, maka RS merupakan radical axis dari ω_1 dan ω_2. Karena $\angle BPS = \angle PRS$, maka BP menyinggung ω_1. Dengan kata lain, $\overline{AB}$ menyingggung ω_1. Secara analog, AC menyinggung ω_2. Maka berlaku

$$\text{Pow}_{\omega_1}(A) = AP^2 \quad \text{dan} \quad \text{Pow}_{\omega_2}(A) = AQ^2.$$

Karena $AP = AQ$, maka $\text{Pow}_{\omega_1}(A) = \text{Pow}_{\omega_2}(A)$. Artinya, A berada di radical axis ω_1 dan ω_2. Kontradiksi bahwa A berada di garis RS. Jadi, $\omega_1 = \omega_2$ seperti yang ingin dibuktikan.

6. Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.

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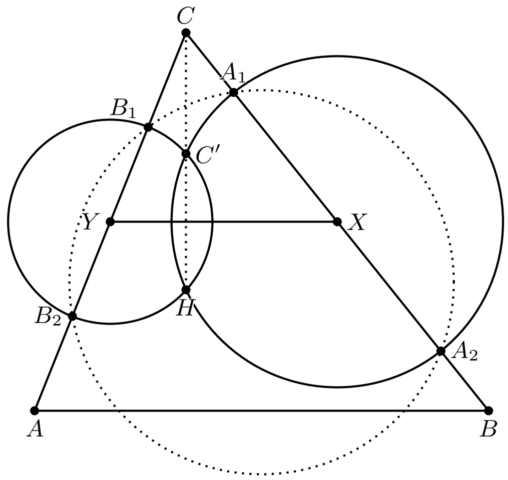

Misalkan $\Gamma_A \cap \Gamma_B \in \{H, C'\}$. Misalkan pula X dan Y berturut-turut merupakan pusat Γ_A dan Γ_B. Dari **Midpoint Theorem**, maka $XY \parallel AB$. Karena $CH \perp AB$, akibatnya $CH \perp XY$. Kita tahu bahwa radical axis Γ_A dan Γ_B akan tegak lurus dengan XY. Dengan kata lain, $XY \perp C'H$. Kita simpulkan bahwa C, C', H segaris. Sehingga kita peroleh

$$CB_1 \cdot CB_2 = CC' \cdot CH = CA_1 \cdot CA_2 \implies CB_1 \cdot CB_2 = CA_1 \cdot CA_2.$$

Maka A_1, A_2, B_1, B_2 siklis. Secara analog, A_1, A_2, C_1, C_2 siklis yang berarti kita simpulkan $A_1, A_2, B_1, B_2, C_1, C_2$ siklis.

7. Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$ respectively are concurrent.

   USA Mathematical Olympiad 1997/Problem 2

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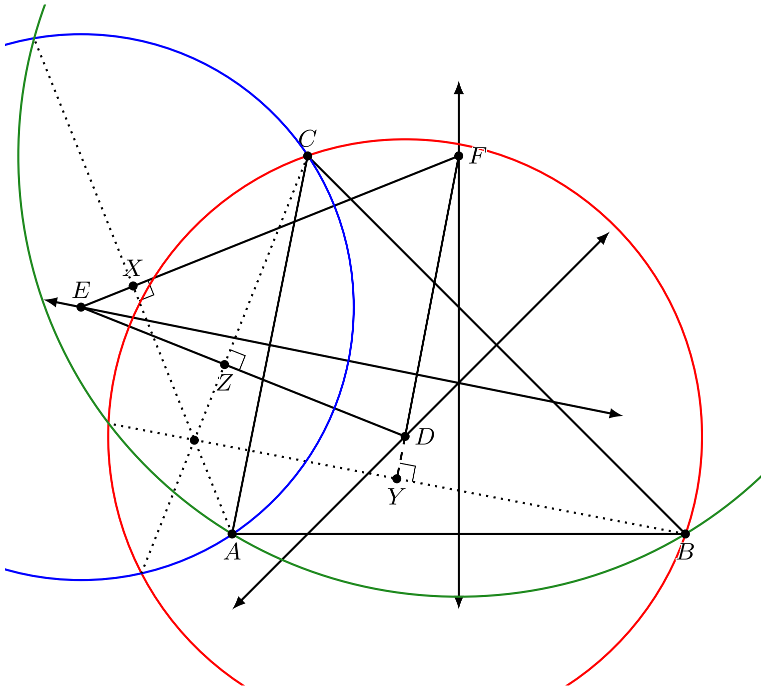
   Misalkan X, Y, Z berturut-turut terletak pada garis EF, DF, DE sehingga $AE \perp EF$, $BY \perp FD$, $CZ \perp DE$.



   Kita tahu bahwa perpotongan ketiga garis sumbu merupakan titik pusat lingkaran luar $\triangle ABC$. Karena D, E, F berada pada garis sumbu $\triangle ABC$, maka lingkaran yang berpusat di D, E, F akan melalui dua dari tiga titik A, B, C. Setiap dua pasang dari ketiga lingkaran tersebut memiliki radical axis yang melalui titik A, B, C. Kita tahu bahwa radical axis akan tegak lurus terhadap garis yang menghubungkan kedua pusat lingkaran tersebut. Karena $EF \perp AX$, kita simpulkan bahwa AX merupakan radical axis dari kedua lingkaran yang berpusat di E dan F. Secara analog, BY dan CZ juga merupakan radical axis dari dua pasang lingkaran. Dari **Radical Axis Theorem**, kita simpulkan AX, BY, CZ akan berpotongan di satu titik.

   Remark. Ketika selesai membaca soal ini, saya terpikir ide dari soal ini adalah radical axis. Masalahnya, lingkaran mana saja yang akan saya gunakan untuk menggunakan Radical Axis Theorem. Saya geser titik D sepanjang garis sumbu $\overline{BC}$ hingga akhirnya mulai terpikir untuk menggambar lingkaran yang berpusat di D dengan DB, DC sebagai jari-jarinya. Soal ini merupakan soal favorit saya di bab ini.

8. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter $\overline{BD}$ at B and N. Prove that the lines AM, DN, XY are concurrent.

International Mathematical Olympiad 1995/Problem 1

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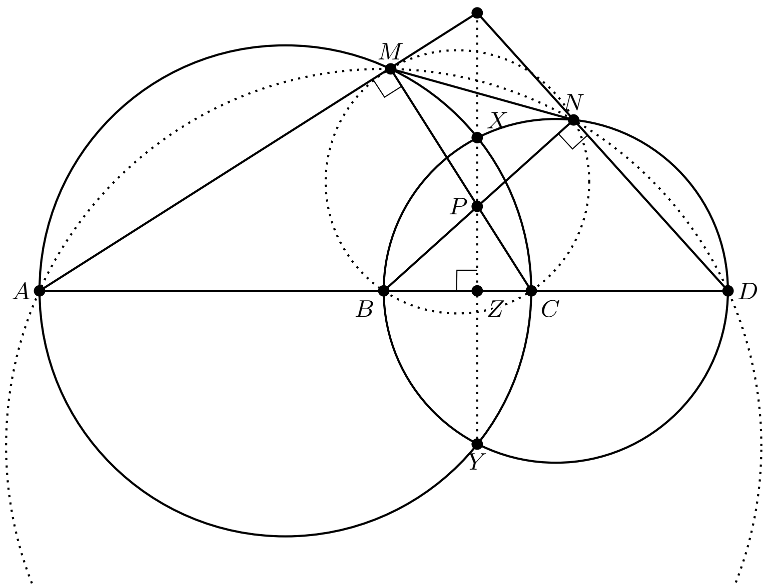

Tinjau bahwa

$$BP \cdot PN = XP \cdot PY = CP \cdot PM \implies BP \cdot PN = CP \cdot PM.$$

Maka $BCNM$ siklis. Tinjau pula

$$\angle DAM = \angle CAM = 90^\circ - \angle ACM = 90^\circ - \angle BCM = 90^\circ - \angle BNM.$$

Kita punya

$$\angle DAM + \angle MND = 90^\circ - \angle BNM + \angle BNM + 90^\circ = 180^\circ \implies \angle DAM + \angle MND = 180^\circ.$$

Maka $AMND$ siklis. Tinjau bahwa AM merupakan radical axis dari $(AMND)$ dan (ACM), ND merupakan radical axis dari $(AMND)$ dan (BND), dan XY merupakan radical axis dari (ACM) dan (BND). Dari **Radical Axis Theorem**, maka AM, DN, XY berpotongan di satu titik.

9. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

USA Mathematical Olympiad 1998/Problem 2

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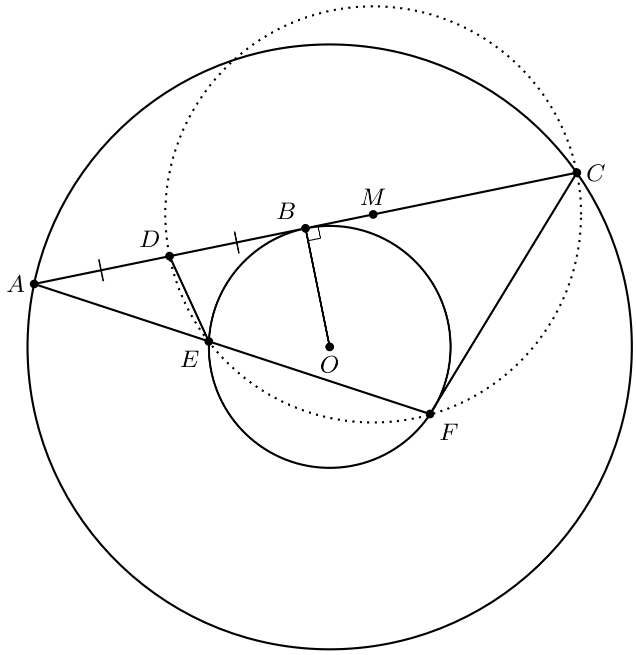
Jawabannya adalah $\boxed{\frac{5}{3}}$.



Misal O pusat kedua lingkaran. Akan kita buktikan bahwa $DEFC$ siklis. Misalkan panjang $AC = 4x$. Tinjau $OB \perp AC$, maka OB adalah apotema $\mathcal{C}_1$ sehingga panjang $AB = BC = 2x$. Karena D titik tengah AB, maka panjang $AD = DB = x$. Perhatikan

$$\text{Pow}_{\mathcal{C}_2}(A) = AB^2 = AE \cdot AF \implies 4x^2 = AE \cdot AF.$$

Karena $AD \cdot AC = x \cdot 4x = 4x^2$, maka $AD \cdot AC = AE \cdot AF$. Sehingga $DEFC$ siklis. Karena M adalah perpotongan dari kedua garis sumbu DE dan FC, maka M adalah titik pusat lingkaran $(DEFC)$. Kita tahu panjang $DC = AC - AD = 4x - x = 3x$. Maka $DM = DC = \frac{3x}{2}$ dan kita peroleh

$$\frac{AM}{MC} = \frac{AD + DM}{MC} = \frac{x + \frac{3x}{2}}{\frac{3x}{2}} = \frac{\frac{5x}{2}}{\frac{3x}{2}} = \frac{5}{3}.$$

10. Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

International Mathematical Olympiad 2000/Problem 1

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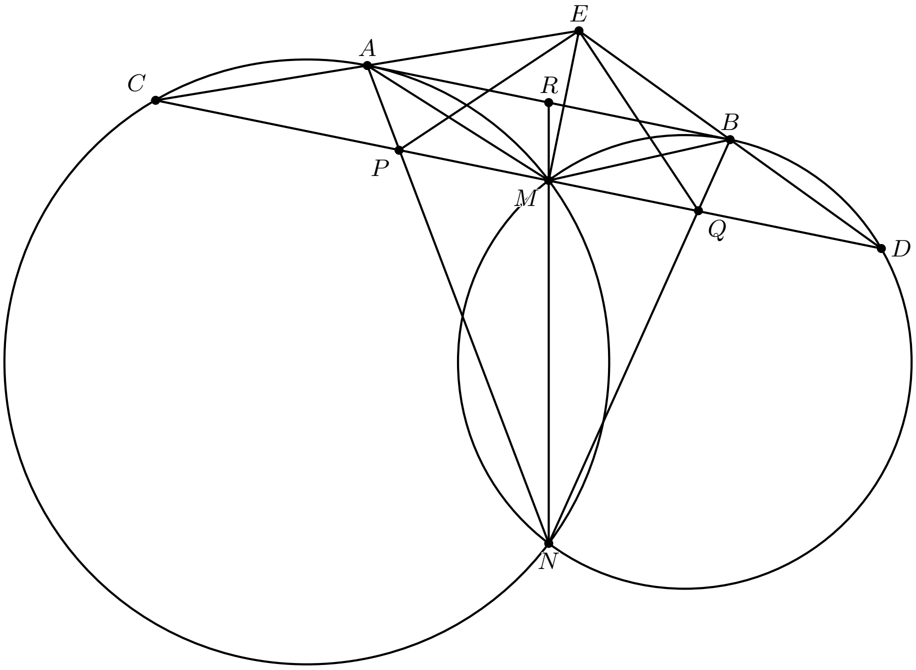
Misalkan $R = MN \cap AB$.



Karena AB menyinggung G_1 di A, maka $\angle BAM = \angle ACM = \angle ANM$. Secara analog, kita peroleh $\angle MBA = \angle MDB = \angle MNB$. Karena $AB \parallel CD$, maka $\angle ACM = \angle EAB$, $\angle BAM = \angle AMC$, $\angle MDB = \angle ABE$, dan $\angle ABM = \angle DMB$. Perhatikan bahwa

$$RA^2 = RM \cdot RN = RB^2 \implies RA^2 = RB^2 \iff RA = RB.$$

Karena $AB \parallel PQ$, maka $\angle NPM = \angle NAR$ dan $\angle PNM = \angle ANR$ sehingga berakibat $\triangle NMP \sim \triangle NRA$. Maka $\frac{MP}{RA} = \frac{NP}{NA}$. Secara analog, diperoleh $\triangle NMQ \sim \triangle NRB$ dan $\triangle NPQ \sim \triangle NAB$. Maka $\frac{MQ}{RB} = \frac{NQ}{NB}$ dan $\frac{NP}{NA} = \frac{NQ}{NB}$. Kita punya

$$\frac{MP}{RA} = \frac{NP}{NA} = \frac{NQ}{NB} = \frac{MQ}{RB} \implies \frac{MP}{RA} = \frac{MQ}{RB}.$$

Maka panjang $MP = MQ$. Tinjau bahwa $\angle MBA = \angle EBA$, panjang $AB = AB$, dan $\angle MAB = \angle EAB$, maka $\triangle MAB \cong \triangle EAB$ (sudut-sisi-sudut). Maka panjang $AM = AE$ dan berakibat

$$\angle PME = \angle PMA + \angle AME = \angle AMC + 90^\circ - \frac{\angle MAE}{2} = \angle AMC + 90^\circ - \angle MAB = 90^\circ.$$

Maka $EM \perp PQ$ sehingga berakibat panjang $EP = EQ$.

11. Let $ABCD$ be a cyclic quadrilateral whose diagonals meet at P. Let W, X, Y, Z be the feet of P onto $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$, respectively. Show that $WX + YZ = XY + WZ$.

Canada 1990/Problem 3

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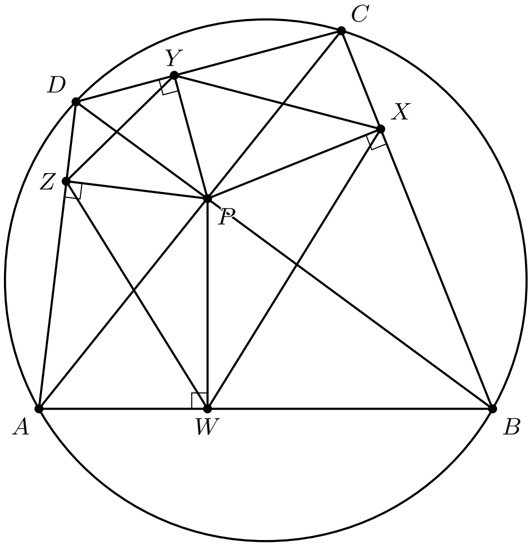

Perhatikan bahwa $\angle AWP + \angle AZP = 180°$, maka $AWPZ$ siklis. Secara analog, $BWPX, CYPX$, dan $PYDZ$ siklis. Kita punya

$$\angle PWZ = \angle PAZ = \angle CAD = \angle CBD = \angle XBP = \angle XWP.$$

Maka $\angle PWZ = \angle PWX$ sehingga PW garis bagi $\angle XWZ$. Secara analog, PX, PY, dan PZ berturut-turut merupakan garis bagi $\angle WXY, \angle XYZ$, dan $\angle YZW$. Maka titik P merupakan titik pusat lingkaran dalam $WXYZ$ (lingkaran yang menyinggung keempat sisi $WXYZ$). Dari **Pitot Theorem**, berlaku $WX + YZ = XY + WZ$.

> **Remark.** Melihat kondisi $WX + YZ = XY + WZ$ mirip-mirip dengan Pitot Theorem. Oleh karena itu, saya mungkin bisa membuktikan terdapat lingkaran dalam yang menyinggung keempat sisi $WXYZ$.

12. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

International Mathematical Olympiad 2009/Problem 2

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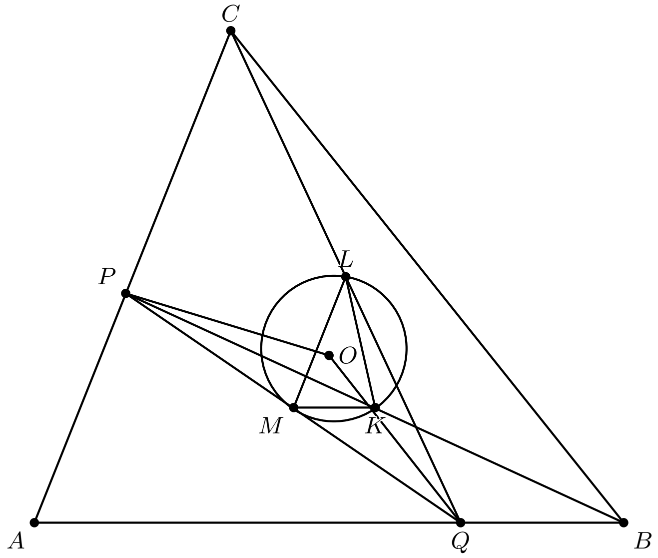

Dari **Midpoint Theorem**, maka $MK \parallel QB$ dan $ML \parallel PC$. Karena PQ menyinggung Γ, maka

$$\measuredangle MLK = \measuredangle QMK = \measuredangle MAQ = \measuredangle PQA \implies \measuredangle MLK = \measuredangle PQA.$$

Secara analog, diperoleh $\measuredangle MKL = \measuredangle QPA$. Maka $\triangle AQP \sim \triangle MLK$. Maka $\frac{AQ}{AP} = \frac{ML}{MK} \iff AP = \frac{AQ \cdot MK}{ML}$. Tinjau $\text{Pow}_{(ABC)}(Q) = QM^2 = QO^2 - R^2$ di mana R panjang jari-jari lingkaran (ABC). Selain itu, $\text{Pow}_{(ABC)}(P) = PO^2 - R^2$. Karena $ML \parallel PC$ yang berakibat $\angle MLQ = \angle PCQ$ dan $\angle QML = \angle QPC$, maka $\triangle QML \sim \triangle QPC$. Maka $\frac{ML}{PC} = \frac{QM}{QP} = \frac{1}{2}$. Secara analog, $\triangle PMK \sim \triangle PQB$ sehingga $\frac{MK}{QB} = \frac{PM}{PQ} = \frac{1}{2}$. Maka

$$AP \cdot PC = \frac{AQ \cdot MK}{ML} \cdot PC = AQ \cdot \frac{MK}{ML} \cdot PC = AQ \cdot \frac{\frac{1}{2}QB}{\frac{1}{2}PC} \cdot PC = AQ \cdot QB.$$

Tinjau bahwa $AP \cdot PC = \text{Pow}_{(ABC)}(P)$ dan $AQ \cdot QB = \text{Pow}_{(ABC)}(Q)$. Sehingga

$$\text{Pow}_{(ABC)}(P) = AP \cdot PC = AQ \cdot QB = \text{Pow}_{(ABC)}(Q) \implies \text{Pow}_{(ABC)}(P) = \text{Pow}_{(ABC)}(Q)$$

yang berarti $PO^2 - R^2 = QO^2 - R^2 \iff OP = OQ$.

13. Let $\overline{AD}, \overline{BE}, \overline{CF}$ be the altitudes of a scalene triangle ABC with circumcenter O. Prove that $(AOD), (BOE)$, and (COF) intersect at point X other than O.

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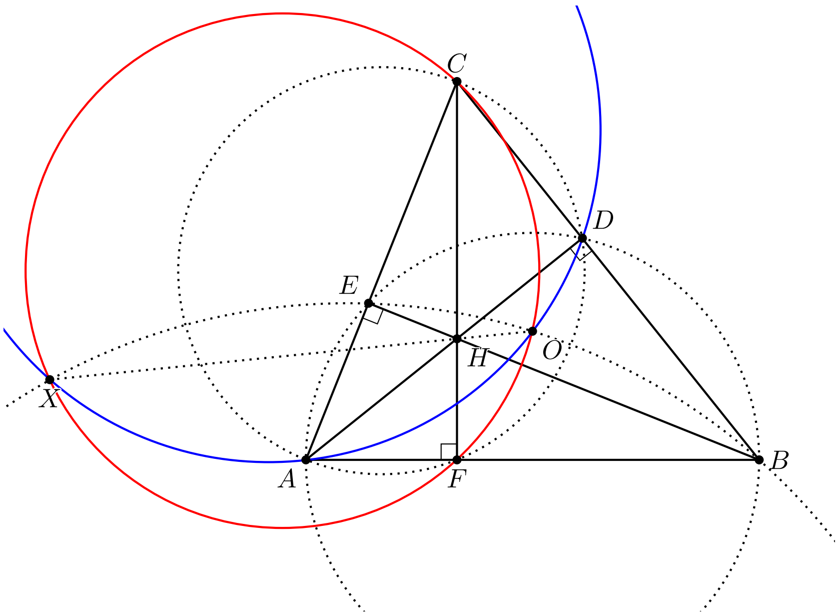
Misalkan H adalah perpotongan ketiga garis tinggi $\triangle ABC$.



Misalkan $X = (AOD) \cap (COF)$ dan kita ingin membuktikan $BOEX$ siklis. Perhatikan bahwa OX merupakan radical axis dari (AOD) dan (COF). Karena $\measuredangle AFC = \measuredangle ADC$, maka $AFDC$ siklis. Kita punya CF adalah radical axis dari (COF) dan $(AFDC)$. Selain itu, AD merupakan radical axis dari (AOD) dan $(AFDC)$. Dari **Radical Axis Theorem**, berakibat AD, CF, OX berpotongan di satu titik H. Perhatikan bahwa $\measuredangle AEB = \measuredangle ADB$, maka $AEDB$ siklis. Sehingga berlaku $AH \cdot HD = BH \cdot HE$. Kita punya

$$OH \cdot HX = AH \cdot HD = BH \cdot HE \implies OH \cdot HX = BH \cdot HE$$

yang berakibat $BOEX$ siklis.

14. Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let $\omega, \omega_1, \omega_2$, and ω_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.

    (a) Prove that ω_1, ω_2, and ω_3 intersect in a common point.

    (b) Show that lines PD, QE, and RF are concurrent.

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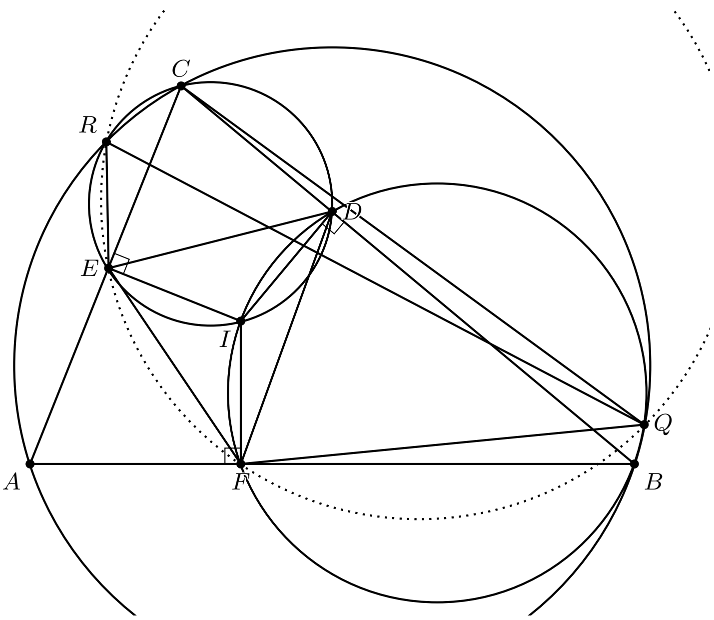
Misalkan I dan O berturut-turut adalah titik bagi dan titik pusat lingkaran luar $\triangle ABC$. Maka $\angle IFA = \angle IDB = \angle IEC = 90°$.



(a) Karena $\angle AFI + \angle AEI = 180°$, maka $AFIE$ siklis. Secara analog, $BFID$ dan $CDIE$ siklis. Maka ω_1, ω_2, dan ω_3 berpotongan di satu titik I.

(b) Kita klaim $REFQ$ siklis. Tinjau

$$\measuredangle QRE = \measuredangle CRE - \measuredangle CRQ = \measuredangle CRE + \measuredangle QRC = \measuredangle CDE + \measuredangle QBC.$$

Misalkan $\angle A = 2a, \angle B = 2b$, dan $\angle C = 2c$, maka $\angle A + \angle B + \angle C = 180° \iff a + b + c = 90°$. Karena panjang $CI = CI, \angle IDC = \angle IEC$, dan panjang $ID = IE$, maka $\triangle IDC \cong \triangle IEC$ (sisi-sudut-sisi). Maka panjang $CD = CE \iff \angle CDE = \angle CED = 90° - c = a + b$. Secara analog, kita punya $\angle BFD = 90° - b \implies \angle DFI = b$ dan $\angle AEF = 90° - a \implies \angle IFE = a$. Maka $\angle EDC = \angle EFD$. Maka

$$\measuredangle CDE + \measuredangle QBC = \measuredangle DFE + \measuredangle QBD = \measuredangle DFE + \measuredangle QFD = \measuredangle QFE.$$

Maka $\measuredangle QRE = \measuredangle QFE$ sehingga $QFER$ siklis. Secara analog, $PEDQ$ dan $PFDR$ siklis. Tinjau PD, QE, RF berturut-turut merupakan radical axis dari pasangan $((PEDQ), (PFDR)), ((PEDQ), (QFER)), ((QEFR), (PFDR))$. Menurut **Radical Axis Theorem**, maka PD, QE, RF berpotongan di satu titik.

15. In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of $\overline{BC}$ and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MF}$, respectively, and let T be on line KL such that $\overline{TA} \parallel \overline{BC}$. Prove that $TA = TM$.

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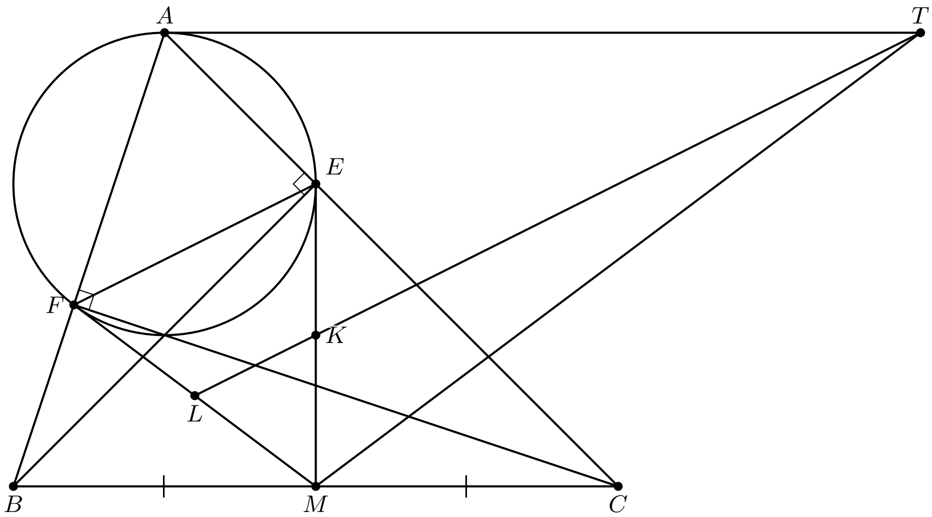

Di *Chapter 1*, kita telah membuktikan ME, MF, TA merupakan garis singgung (AEF) serta M merupakan pusat $(BFEC)$. Kita punya $\text{Pow}_{(AEF)} = LF^2$. Kontruksi lingkaran berjari-jari nol yang berpusat di M, katakan ω. Kita punya $\text{Pow}_\omega(L) = ML^2$. Karena $ML = LF$, maka $\text{Pow}_{(AEF)}(L) = \text{Pow}_\omega(L)$. Dengan cara yang sama, diperoleh $\text{Pow}_{(AEF)}(K) = \text{Pow}_\omega(K)$. Maka LK merupakan radical axis (AEF) dan ω. Karena T berada di LK, berlaku

$$\text{Pow}_{(AEF)}(T) = \text{Pow}_\omega(T) \implies TA^2 = TM^2 \iff TA = TM.$$

Remark. Ketika ingin membuktikan $TA = TM$, kemudian saya *notice* bahwa $\text{Pow}_{(AEF)}(T) = TA^2$, maka dari itu saya termotivasi untuk membuktikan garis LK merupakan radical axis dari (AEF) dan suatu lingkaran. Dengan kata lain, saya harus menemukan suatu lingkaran ω yang berakibat $\text{Pow}_{(AEF)}(T) = \text{Pow}_\omega(T)$. Karena panjang $LM = LF$, saya mulai bisa menemukan radical axis dan lingkaran mana yang bisa digunakan. Jika saya ambil $\text{Pow}_\omega(T) = TM^2$, hal ini membuat saya termotivasi untuk membuat lingkaran berjari-jari nol (setelah melihat contoh All-Russian MO 2010).